

A POINTWISE FINITE-DIMENSIONAL REDUCTION METHOD FOR A FULLY COUPLED SYSTEM OF EINSTEIN-LICHTNEROWICZ TYPE.

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ABSTRACT. We construct non-compactness examples for the fully coupled Einstein-Lichnerowicz constraint system in the focusing case. The construction is obtained by combining pointwise *a priori* asymptotic analysis techniques, finite-dimensional reductions and a fixed-point argument.

More precisely, we perform a fixed-point procedure on the remainders of the expected blow-up decomposition. The argument consists of an involved finite-dimensional reduction coupled with a ping-pong method. To overcome the non-variational structure of the system, we work with remainders which belong to strong function spaces and not merely to energy spaces. Performing both the ping-pong argument and the finite-dimensional reduction therefore heavily relies on the *a priori* pointwise asymptotic techniques of the C^0 theory, as developed in Druet-Hebey-Robert [18].

1. INTRODUCTION

1.1. Statement of the results. Let (M, g) be a closed Riemannian manifold of dimension $n \geq 6$, where closed means here compact without boundary. We investigate non-compactness issues in strong spaces for the set of positive solutions of the Einstein-Lichnerowicz system of equations in M :

$$\begin{cases} \Delta_g u + hu = fu^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} \\ \vec{\Delta}_g T = u^{2^*} X + Y. \end{cases} \quad (1.1)$$

The unknowns of (1.1) are u , a smooth positive function in M , and T , a smooth field of 1-forms in M . In (1.1) we denoted by $\mathcal{L}_g T$ the conformal Killing derivative of T , whose expression in coordinates is:

$$\mathcal{L}_g T_{ij} = \nabla_i T_j + \nabla_j T_i - \frac{2}{n} (\operatorname{div}_g T) g_{ij}, \quad (1.2)$$

and we denoted by $\vec{\Delta}_g$ the Lamé operator acting on 1-forms as $\vec{\Delta}_g T = -\operatorname{div}_g(\mathcal{L}_g T)$. In (1.2), ∇ stands for the Levi-Civita connection of the metric. The second equation of (1.1) is in particular a 1-form equation. Also, in (1.1), $\Delta_g = -\operatorname{div}_g(\nabla \cdot)$ is the Laplace-Beltrami operator, h, f, π are smooth functions in M , σ is a smooth field of 2-forms with $\operatorname{tr}_g \sigma = 0$ and $\operatorname{div}_g \sigma = 0$ and X and Y are smooth fields of 1-forms in M . The notation $2^* = \frac{2n}{n-2}$ denotes the critical exponent for the embedding of the Sobolev space $H^1(M)$ into Lebesgue spaces. We also assume that:

$$f > 0 \text{ in } M, \quad (1.3)$$

and that $\Delta_g + h$ is coercive (which is necessary in view of (1.3)).

The Einstein-Lichnerowicz system (1.1) arises in the initial-value problem in Mathematical General Relativity, as a conformal formulation of the constraint equations; see Bartnik-Isenberg [2] for the physical derivation of (1.1). In the relativistic physical case, the coefficients of (1.1) express in terms of the background physics data of the problem: for instance, in a scalar-field setting, they write as:

$$h = c_n (S_g - |\nabla \psi|_g^2), \quad f = c_n \left(2V(\psi) - \frac{n-1}{n} \tau^2 \right), \quad X = -\frac{n-1}{n} \nabla \tau, \quad Y = -\pi \nabla \psi, \quad (1.4)$$

where $c_n = \frac{n-2}{4(n-1)}$. In (1.4), $V : \mathbb{R} \rightarrow \mathbb{R}$ is a potential, $\psi, \pi : M \rightarrow \mathbb{R}$ are scalar-field components, $\tau : M \rightarrow \mathbb{R}$ is the mean extrinsic curvature and S_g is the scalar curvature of g . In view of (1.4), the focusing assumption (1.3) naturally arises as one considers non-trivial non-gravitational physics data.

In the focusing case (1.3), existence results for (1.1) were first obtained in Hebey-Pacard-Pollack [26], under the assumption $X \equiv 0$ that decouples (1.1); multiplicity results were then obtained in Ma-Wei [35], Premoselli [38] and, for specific physical cases, in Holst-Meier [30], Chruściel-Gicquaud [9] and Bizoń-Pletka-Simon [6]. For the fully coupled system – when $X \not\equiv 0$ – existence results are in Premoselli [37] and Gicquaud-Nguyen [24].

A more satisfactory picture of system (1.1) is obtained through the analysis of its stability issues. Following the general definition given in Druet [15] (see also Hebey [25]), we say that system (1.1) is *stable* if, for any sequence $(h_k, f_k, \pi_k, \sigma_k, X_k, Y_k)_k$ of coefficients converging towards $(h, f, \pi, \sigma, X, Y)$ as $k \rightarrow +\infty$ in some strong topology (to be precised), and for any sequence $(u_k, T_k)_k$ of solutions of:

$$\begin{cases} \Delta_g u_k + h_k u_k = f_k u_k^{2^*-1} + \frac{|\mathcal{L}_g T_k + \sigma_k|_g^2 + \pi_k^2}{u_k^{2^*+1}} \\ \vec{\Delta}_g T_k = u_k^{2^*} X_k + Y_k, \end{cases} \quad (1.5)$$

with $u_k > 0$, there holds, up to a subsequence and up to elements in the kernel of \mathcal{L}_g , that $(u_k, T_k)_k$ converges to some positive solution (u_0, T_0) of (1.1) in $C^{1,\eta}(M)$ for all $0 < \eta < 1$. One defines the *compactness* of (1.1) analogously, by taking a constant sequence of coefficients $(h_k, f_k, \pi_k, \sigma_k, X_k, Y_k)_k = (h, f, \pi, \sigma, X, Y)$. In the focusing case (1.3) and for the decoupled system (when $X \equiv 0$), stability results were first obtained in Druet-Hebey [16], and then in Hebey-Veronelli [28] and Premoselli [38]. For the fully coupled case $X \not\equiv 0$, the stability of (1.1) has been investigated in Druet-Premoselli [21] and Premoselli [39] on locally conformally flat manifolds. It is shown in [39] that for any $n \geq 6$, equation (1.1) is stable in the C^2 topology as soon as $\pi \not\equiv 0$ and:

$$\text{either } X \text{ and } \nabla f \text{ have no common zero in } M \quad \text{or} \quad h - c_n S_g + \frac{(n-2)(n-4)}{8(n-1)} \frac{\Delta_g f}{f} < 0, \quad (1.6)$$

where S_g denotes the scalar curvature of g . Stability actually holds for a slightly weaker topology, see [21] and [39]. The stability of the decoupled version of (1.1) was used in [38] to describe multiplicity issues in small dimensions. In the physical case, where the coefficients are given by (1.4), the stability and the instability of (1.1) relate to the relevance of the conformal method from which it arises. We refer to the discussion in [39]. In the opposite direction, the only known instability result for (1.1) has been obtained by Premoselli-Wei [40] in the decoupled case $X \equiv 0$, and for the physical choice of coefficients given by (1.4). The motivation in [40] was to obtain explicit physical counter-examples to the stability of the conformal method.

In this work, we address the instability behavior of system (1.1) from a different perspective: we leave the physical case treated in [40] aside but consider instead the fully coupled case $X \not\equiv 0$ of (1.1). As a direct counterpart of the results in [21] and [39], we obtain non-compactness results for system (1.1) in any dimension $n \geq 6$ as soon as (1.6) is not satisfied. Our main result states as follows:

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 6$ of positive Yamabe type and possessing no non-trivial conformal Killing fields. There exist regular coefficients $(h, f, \pi, \sigma, X, Y)$, with $\Delta_g + h$ coercive, $f > 0$, $\pi \not\equiv 0$ and $X \not\equiv 0$ such that the associated system of equations (1.1):*

$$\begin{cases} \Delta_g u + hu = fu^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} \\ \vec{\Delta}_g T = u^{2^*} X + Y \end{cases}$$

possesses a blowing-up sequence of solutions $(u_k, T_k)_k$, that is satisfying $\|u_k\|_{L^\infty(M)} \rightarrow +\infty$ and $\|\mathcal{L}_g T_k\|_{L^\infty(M)} \rightarrow +\infty$ as $k \rightarrow +\infty$. Also, the u_k are positive, possess a single blow-up point and blow-up with a non-zero limit profile.

Theorem 1.1 is a *non-compactness* result, and therefore also an instability result. The explicit expression of the coefficients $(h, f, \pi, \sigma, X, Y)$, their regularity as well as the generality of the construction are discussed in Section 2 below. For $n \geq 7$, these coefficients do not satisfy (1.6), thus establishing its sharpness in the coupled case. A conformal Killing field is a field of 1-forms X in M satisfying $\mathcal{L}_g X = 0$. The assumption that (M, g) possesses no non-trivial conformal Killing fields is generic as shown in Beig-Chruściel-Schoen [3] and implies, since M is closed, that $\vec{\Delta}_g$ has no kernel.

A striking consequence of Theorem 1.1 is the existence of an infinite number of solutions of (1.1):

Corollary 1.2. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 6$ of positive Yamabe type and possessing no non-trivial conformal Killing fields. There exist regular coefficients $(h, f, \pi, \sigma, X, Y)$, with $\Delta_g + h$ coercive, $f > 0$, $\pi \not\equiv 0$ and $X \not\equiv 0$ such that the system (1.1) possesses an infinite number of solutions.*

In the fully coupled case case $X \not\equiv 0$ that we investigate here, system (1.1) is of a different nature than in the physical decoupled case of [40]. In particular, (1.1) possesses no variational structure, exhibits supercritical nonlinearities and is not well-posed for (u, W) in the energy space $H^1(M)$. To prove Theorem 1.1 we thus develop a pointwise finite-dimensional reduction method, which combines a priori pointwise asymptotic analysis techniques with a variational Lyapunov-Schmidt-type approach in order to perform an involved ping-pong method in strong spaces. We explain in detail the method and the strategy of proof in the next sub-section.

1.2. Strategy of the proof of Theorem 1.1. In the fully coupled case $X \not\equiv 0$, system (1.1) exhibits a strong nonlinear coupling via the $(|\mathcal{L}_g T + \sigma|_g^2 + \pi^2)u^{-2^*-1}$ term, as well as a super-critical nonlinearity in the right-hand side of the 1-form equation. Hence (1.1) does not possess a well-posed variational formulation in $H^1(M)$. If one only assumes that $u \in H^1(M)$, the right-hand side of the 1-form equation is merely bounded in $L^1(M)$ which yields no integral control on $(|\mathcal{L}_g T + \sigma|_g^2 + \pi^2)u^{-2^*-1}$ whatsoever. This is a serious obstacle to combining a ping-pong approach with a finite-dimensional reduction method for the scalar equation of (1.1).

To prove Theorem 1.1 we therefore work in strong topologies. We construct a blowing-up sequence of solutions $(u_k, T_k)_k$ of (1.1) whose scalar component writes as:

$$u_k = u_{k,t,p} = W_{k,t,p} + u + \varphi_{k,t,p}, \quad (1.7)$$

but where this decomposition holds in $C^0(M)$ and not in $H^1(M)$. Here, as usual, $W_{k,t,p}$ denotes a bubbling profile depending on some parameters $(t, p) \in \mathbb{R}^{n+1}$ whose expression is given in Sections 2 and 7 below. The positive function u is the scalar component of a solution (u, T) of (1.1) satisfying a non-degeneracy assumption (detailed in (2.13) below). And the remainder term, $\varphi_{k,t,p}$, is chosen to be small with respect to $W_{k,t,p}$ and u in the following pointwise sense:

$$|\varphi_{k,t,p}| \leq \varepsilon_k (W_{k,t,p} + u) \quad \text{pointwise in } M, \quad (1.8)$$

for some suitably chosen sequence $(\varepsilon_k)_k$ of positive numbers converging to zero.

Decomposition (1.7) is of course reminiscent of Struwe's H^1 a priori result [47]. But the motivation for the choice of (1.7) comes from the work of Druet-Hebey-Robert [18], where Struwe's decomposition was shown to actually hold true in $C^0(M)$ for H^1 -bounded solutions of critical stationary Schrödinger equations. The blow-up analysis of system (1.1) performed in [21] and [39] confirms the choice of (1.7) *pointwise*, at least at a local scale.

The construction of a sequence $(u_k, T_k)_k$ satisfying (1.7) goes through an involved ping-pong procedure which relies upon a finite-dimensional reduction made possible by asymptotic analysis techniques. Section 2 introduces the definition of the coefficients $(h, f, \pi, \sigma, X, Y)$ and of the bubbling profiles considered and Section 9 gathers several technical results used throughout the article. The structure of the proof and the organization of the remaining sections of the article are as follows:

Section 3: Semi-decoupling and H^1 reduction. Let φ be a remainder satisfying (1.8). Since $\vec{\Delta}_g$ has no Kernel by assumption, there exists a unique 1-form $T_{k,t,p}$ in M satisfying $\vec{\Delta}_g T_{k,t,p} = (W_{k,t,p} + u + \varphi)^{2^*} X + Y$. The C^0 bound (1.8) on φ yields explicit pointwise bounds on $\mathcal{L}_g T_{k,t,p}$, which happens to blow-up too fast for standard H^1 finite-dimensional reduction procedures to apply to the scalar equation of (1.1) with $\mathcal{L}_g T_{k,t,p}$ seen as a coefficient. We therefore artificially discard the $|\mathcal{L}_g T_k + \sigma|_g^2$ term into a source term and consider instead the following equation:

$$\Delta_g u + hu = fu^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} + \underbrace{\left(\frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(W_{k,t,p} + u + \varphi)^{2^*+1}} \right)}_{\text{Source term}}, \quad (1.9)$$

where T satisfies $\vec{\Delta}_g T = u^{2^*} X + Y$. We perform a standard finite-dimensional reduction procedure and construct a solution of (1.9) – up to Kernel elements – having the form

$$u_{k,t,p} = W_{k,t,p} + u + \psi, \quad (1.10)$$

for some new remainder $\psi \in H^1(M)$.

Sections 4, 5 and 6: Pointwise asymptotic analysis of the remainder ψ and fixed-point argument. In view of the ping-pong argument, the main part of the proof is an involved application of Banach-Picard's fixed-point theorem to the remainders' mapping $\varphi \mapsto \psi^1$ in the strong space of functions satisfying (1.8). The finite-dimensional reduction provides an H^1 bound on ψ but says nothing about a C^0 control. In this part we show that ψ still satisfies the pointwise estimate (1.8). A first step consists in showing that

$$\psi = o(W_{k,t,p} + u) \text{ in } C^0(M) \quad (1.11)$$

as $k \rightarrow +\infty$. This step is achieved in Section 4 through an asymptotic blow-up analysis of the solution $u_{k,t,p}$ in (1.10) and relies on the machinery of the C^0 theory, developed in Druet-Hebey-Robert [18] (see also Hebey [25]), adapted to take into account the source term in (1.9).

A second step consists in quantifying the $o(1)$ in (1.11). This is the purpose of Section 5. Proposition 5.1 provides a local improvement of (1.11) in the region where the bubbling profile is dominant in the C^0 decomposition of $u_{k,t,p}$. It is obtained by a second-order blow-up analysis on and relies on the fact that, by construction, ψ is orthogonal to the Kernel elements of the finite-dimensional reduction. The global improved version of (1.11) is Proposition 5.3, which yields the following control on ψ : for some positive constant C independent of k, t, p ,

$$|\psi| \leq C(\|X\|_{L^\infty} \varepsilon_k + \mu_k)(W_{k,t,p} + u).$$

The latter estimate implies in particular that, provided $\|X\|_{L^\infty(M)}$ is sufficiently small, ψ satisfies again (1.8) for a suitable choice of $(\varepsilon_k)_k$.

Section 6 contains the last step in the proof of the fixed-point argument. The application of Banach-Picard's fixed-point theorem is performed in Proposition 6.1 using again asymptotic analysis techniques in the spirit of those appearing in the proof of Proposition 5.1. In Proposition 6.3 we also

¹we use the notations of the previous paragraph

obtain an improvement of the global estimate on the remainder, in the region where the weak limit becomes dominant in the C^0 decomposition of $u_{k,t,p}$.

Sections 4, 5 and 6 are the core of the analysis of the paper. The degree of precision of the local and global pointwise estimates on ψ is also the key tool in the final step of the proof of Theorem 1.1.

Sections 7 and 8: Annihilation of the Kernel components.

Section 6 provides us with a solution $(u_{k,t,p}, W_{k,t,p})_k$, $u_{k,t,p} = W_{k,t,p} + u + \varphi_k(t, p)$, of (1.1) up to kernel elements of the scalar equation:

$$\begin{cases} \Delta_g u_{k,t,p} + h u_{k,t,p} = f u_{k,t,p}^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W_{k,t,p}|_g^2}{u_{k,t,p}^{2^*+1}} + \sum_{j=0}^n \lambda_{k,j}(t, p) Z_{j,k,t,p}, \\ \vec{\Delta}_g W_{k,t,p} = u_{k,t,p}^{2^*} X + Y, \end{cases} \quad (1.12)$$

where the $Z_{j,k,t,p}$ are defined in (2.21) (see also Section 7). The proof is concluded by showing that one can annihilate all the $\lambda_{k,j}(t, p)$, $0 \leq j \leq n$, for a suitable value $(t_k, p_k)_k$ of the parameters.

In Section 7 we compute an asymptotic expansion of the $\lambda_{k,j}(t, p)$ as $k \rightarrow +\infty$. The lack of a variational structure forces us to again proceed differently than in the variational setting. Since the remainder $\varphi_k(t, p)$ of the solution $u_{k,t,p}$ was not obtained by variational means we only possess a rough estimate on $\|\varphi_k(t, p)\|_{H^1(M)}$. It is given by (6.2) below, and is intimately related to the choice of ε_k – and hence to the (limited) precision of (5.56) below. In particular, no matter the precision on the choice of the ansatz $W_{k,t,p} + u$, we cannot hope to get a better estimate on $\|\varphi_k(t, p)\|_{H^1(M)}$. Therefore, the usual characterization of the $\lambda_{k,j}(t, p)$ in terms of a reduced energy does not hold here. We overcome this by estimating the $\lambda_{k,j}(t, p)$ directly, using the second-order pointwise estimates on $\varphi_k(t, p)$ and $\nabla \varphi_k(t, p)$ obtained in Sections 5 and 6. The conclusion of the proof of Theorem 1.1 is then given in Section 8, using a degree-theoretic argument.

Let us conclude this introduction with a few remarks:

- We prove Theorem 1.1 assuming that the L^∞ norm of the coupling field X is small, depending on n, g, h, f, π, σ (see Section 2). This assumption is harmless, since smallness conditions on X are necessary for solutions of (1.1) to exist: see [26, 37, 38].
- Our choice of the coefficients $(h, f, \pi, \sigma, X, Y)$ is driven by the a priori stability analysis of [39]. This is explicit in the choice of the order of smallness of X at bumps, and in the localisation of the concentration point constructed. As shown in [39], the latter point is a common zero of X and ∇f and a zero of $h - \frac{n-2}{4(n-1)} S_g$ (except in dimension 6, see below).
- We chose to construct here *non-compactness* examples, which forces us to work with bubbling profiles supported in shrinking balls (see Section 2). Up to obvious modifications of the coefficients $(h, f, \pi, \sigma, X, Y)$, our proof works for bubbling profiles supported on balls of fixed positive radius thus yielding instability examples for (1.1).
- The scalar component of our blowing-up solutions is constructed to have a positive weak limit u . This is necessary, as shown by Lemma 3.1 below.
- The non-degeneracy assumption on u is crucially used twice in our proof: to achieve the finite-dimensional reduction procedure on equation (1.9) and to prove Proposition 5.3. It does not restrain the choice of the coefficients (h, f, π, σ, Y) : as shown in Section 2, for small $\|X\|_\infty$, (1.1) always possesses such a non-degenerate solution.
- As already noticed in [40], the 6-dimensional case requires a more careful analysis. Roughly speaking, we are forced to perform an expansion of higher order than in dimensions $n \geq 7$, see Section 7 below. In particular, a rescaling of X at a concentration point appears in the limiting expansion, see (7.44) and (7.45) below.

Possible references for the finite-dimensional reduction method alluded to above, without pretending to be exhaustive, are Ambrosetti-Malchiodi [1], Berti-Malchiodi [4], Brendle [7], Brendle-Marques [8], Del Pino-Felmer-Musso [10], Del Pino-Musso-Pacard [11], Del Pino-Musso-Pacard-Pistoia [12], Rey [41], Rey-Wei [42], Robert-Vétois [44, 46], Wei [48, 49, 50, 51], and the references therein.

The a priori analysis techniques used in our proof have been developed in the context of the C^0 theory in Druet-Hebey-Robert [18], see also Druet [13], Druet-Hebey [17], Druet-Hebey-Vétois [20] and Hebey-Robert [27]. Related techniques have independently been developed in the investigation of compactness phenomena for the Yamabe problem (see Li-Zhu [34], Druet [14], Marques [36], Li-Zhang [33], Khuri-Marques-Schoen [31]) and in the investigation of stability issues for nonlinear elliptic equations (Hebey-Wei [29], Druet-Hebey-Vétois [19]).

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2. SETTING OF THE PROBLEM AND NOTATIONS

In what follows we let (M, g) be a closed n -dimensional Riemannian manifold, $n \geq 6$, of positive Yamabe type. We will always assume that (M, g) possesses no non-trivial conformal killing 1-forms or, equivalently, that the operator $\vec{\Delta}_g$ is invertible in M . Let $\xi_0 \in M$ and assume that $|W(\xi_0)|_g > 0$ if (M, g) is not locally conformally flat, where $W(\xi_0)$ denotes the Weyl tensor of g . The standard conformal normal coordinates result of Lee-Parker [32] asserts that there exists $\Lambda \in C^\infty(M \times M)$ such that for any point $\xi \in M$ there holds, for some arbitrarily large integer N :

$$\left| \left(\exp_\xi^{g_\xi} \right)^* g_\xi \right| (y) = 1 + O(|y|^N), \quad (2.1)$$

C^1 -uniformly in $\xi \in M$ and in $y \in T_\xi M$ in a small geodesic ball for the metric g_ξ . In (2.1) we have let

$$g_\xi = \Lambda_\xi^{\frac{4}{n-2}} g, \quad (2.2)$$

where the conformal factor $\Lambda_\xi = \Lambda(\xi, \cdot)$ can in addition be chosen to satisfy:

$$\Lambda_\xi(\xi) = 1, \quad \nabla \Lambda_\xi(\xi) = 0. \quad (2.3)$$

In (2.1), the notation $\exp_\xi^{g_\xi}$ denotes the exponential map for the metric g_ξ at point ξ with the identification of $T_\xi M$ to \mathbb{R}^n via a smooth orthonormal basis (e_1, \dots, e_n) of $T_\xi M$ defined in a neighbourhood of ξ_0 . The Lee-Parker [32] result also assert that there holds, for any $\xi \in M$:

$$S_{g_\xi}(\xi) = 0, \quad \nabla S_{g_\xi}(\xi) = 0, \quad \Delta_{g_\xi} S_{g_\xi}(\xi) = \frac{1}{6} |W_g(\xi)|_g^2, \quad (2.4)$$

where S_{g_ξ} denotes the scalar curvature of the conformal metric g_ξ .

Let $(\tau_k)_k$ be a sequence of positive real numbers such that $\sum_k \tau_k < +\infty$. We define a sequence $(\mu_k)_k$ as follows:

$$\mu_k = \begin{cases} \tau_k & \text{if } n = 6, \\ \tau_k^{\frac{2}{n-6}} & \text{if } (M, g) \text{ is l.c.f. or if } 7 \leq n \leq 10, \\ \tau_k^{\frac{1}{2}} & \text{if } n \geq 11 \text{ and } (M, g) \text{ is not l.c.f.} \end{cases} \quad (2.5)$$

Let $(\xi_k)_k$ be a sequence of points of M converging towards ξ_0 and satisfying $d_g(\xi_k, \xi_{k+1}) < \frac{1}{k^2}$ as $k \rightarrow +\infty$. Let $(\beta_k)_k$ be a sequence of positive numbers converging to zero as $k \rightarrow +\infty$ and satisfying:

$$\begin{aligned} \beta_k &\gg \mu_k \text{ if } n \geq 7 \\ \mu_k^{\frac{1}{2}} &\gg \beta_k \gg \mu_k \text{ if } n = 6. \end{aligned} \quad (2.6)$$

Let f be a smooth positive function, let σ be a smooth traceless and divergence-free $(2, 0)$ -tensor in M and let π be a smooth function in M with $\pi \not\equiv 0$. Let Y be a smooth field of 1-forms and denote by \tilde{Y} the only solution of $\vec{\Delta}_g \tilde{Y} = Y$ in M . We let also H be a smooth nonnegative function in \mathbb{R}^n , compactly supported in $B_0(1)$ with $H(0) = 1$, and for which 0 is a *non-degenerate critical point*.

The $n \geq 7$ case. We define:

$$h = c_n S_g + \sum_{k \geq 0} \tau_k H \left(\frac{1}{\beta_k} (\exp_{\xi_k}^{g_{\xi_k}})^{-1}(x) \right), \quad (2.7)$$

where we have let $c_n = \frac{n-2}{4(n-1)}$ for all n . With this definition, h is equal to $c_n S_g + \tau_k H \left(\frac{1}{\beta_k} (\exp_{\xi_k}^{g_{\xi_k}})^{-1}(\cdot) \right)$ on every $B_{\xi_k}(\beta_k)$ – where the ball is taken for the metric g_{ξ_k} – and equals $c_n S_g$ outside of their reunion. Note, with (2.5) and (2.6) and since we are considering the $n \geq 7$ case here, that for any $r \in \mathbb{N}^*$, one can always choose β_k as in (2.6) so that $h \in C^r(M)$. Let u_0 be a smooth, positive solution of the following Einstein-Lichnerowicz equation:

$$\Delta_g u_0 + h u_0 = f u_0^{2^*-1} + \frac{|\mathcal{L}_g \tilde{Y} + \sigma|_g^2 + \pi^2}{u_0^{2^*+1}}, \quad (2.8)$$

The coefficients f, π, σ and Y can always be chosen so that (2.8) possesses a smooth positive solution, see Hebey-Pacard-Pollack [26]. Up to a slight modification of f, π, σ, Y we can always assume u_0 to be *strictly stable*, that is satisfying, for any $\psi \in H^1(M)$:

$$\int_M |\nabla \psi|_g^2 + \left[c_n S_g - (2^* - 1) f u_0^{2^*-2} + (2^* + 1) \frac{|\mathcal{L}_g \tilde{Y} + \sigma|_g^2 + \pi^2}{u_0^{2^*+2}} \right] \psi^2 dv_g \geq \frac{1}{C_0} \|\psi\|_{H^1}^2, \quad (2.9)$$

for some positive constant C_0 . This is the case if u_0 is chosen to be the smallest solution of (2.8), see Premoselli [38], Section 7. Such a minimal solution is then the only strictly stable one of (2.8) (see Dupaigne [22], proposition 1.3.1).

The $n = 6$ case. Up to assuming that the $(\tau_k)_k$ are small enough, and since (M, g) is of positive Yamabe type, standard sub- and super-solution arguments show that there exists a unique positive solution u_0 of:

$$\Delta_g u_0 + \left(\frac{1}{5} S_g - \sum_{k \geq 0} \tau_k H \left(\frac{1}{\beta_k} (\exp_{\xi_k}^{g_{\xi_k}})^{-1}(x) \right) \right) u_0 = -f u_0^2 + \frac{|\mathcal{L}_g \tilde{Y} + \sigma|_g^2 + \pi^2}{u_0^4}.$$

With (2.6), for any $0 < \alpha < 1$ such a u_0 can be chosen to belong to $C^{3,\alpha}(M)$ and it is then a positive and strictly stable solution (in the sense of (2.9)) of the following Einstein-Lichnerowicz equation:

$$\Delta_g u_0 + h u_0 = f u_0^2 + \frac{|\mathcal{L}_g \tilde{Y} + \sigma|_g^2 + \pi^2}{u_0^4},$$

where we have let in this case:

$$h = \frac{1}{5} S_g + 2f u_0 - \sum_{k \geq 0} \tau_k H \left(\frac{1}{\beta_k} (\exp_{\xi_k}^{g_{\xi_k}})^{-1}(x) \right). \quad (2.10)$$

Again, (2.5) and (2.6) ensure that this h can be chosen to belong to $C^{1,\alpha}(M)$ for fixed $0 < \alpha < 1$, but not to $C^2(M)$, contrary to the $n \geq 7$ case.

Now, in every dimension $n \geq 6$, the implicit function theorem shows that there exists a constant $\eta_0 = \eta_0(n, g, h, f, \pi, \sigma, Y)$ such that, for any field X of 1-forms in M satisfying

$$\|X\|_{L^\infty(M)} = \eta \leq \eta_0, \quad (2.11)$$

the Einstein-Lichnerowicz system of equations:

$$\begin{cases} \Delta_g u + hu = fu^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|^2 + \pi^2}{u^{2^*+1}} \\ \vec{\Delta}_g T = u^{2^*} X + Y, \end{cases} \quad (2.12)$$

possesses a solution $(u(X), T(X))$ such that $u(X) \rightarrow u_0$ in $C^2(M)$ as η , defined in (2.11), goes to 0. In (2.12), h is given by (2.7) or (2.10), depending on the dimension. Up to choosing η small enough, it is easily seen that $u(X)$ is again a strictly stable solution of the scalar equation of (2.12), that is:

$$\int_M |\nabla \psi|_g^2 + \left[h - (2^* - 1)fu(X)^{2^*-2} + (2^* + 1) \frac{|\mathcal{L}_g T(X) + \sigma|_g^2 + \pi^2}{u(X)^{2^*+2}} \right] \psi^2 dv_g \geq \frac{1}{C} \|\psi\|_{H^1}^2, \quad (2.13)$$

for any $\psi \in H^1(M)$ and for some positive constant C independent of η . In the following, for the sake of clarity and since X will be fixed, the dependence on X of the solutions $(u(X), T(X))$ will be omitted, and they will just be denoted by (u, T) .

The space $H^1(M)$ will denote the standard Sobolev space of functions in L^2 possessing L^2 distributional derivatives, and we endow $H^1(M)$ with the following scalar product: for any $u, v \in H^1(M)$,

$$\langle u, v \rangle_h = \int_M (\langle \nabla u, \nabla v \rangle_g + huv) dv_g, \quad (2.14)$$

where h is given by (2.7) or (2.10). For any $J \in H^1(M)'$ we will denote by $(\Delta_g + h)^{-1}(J)$ the unique element of $H^1(M)$ such that for any $v \in H^1(M)$ there holds:

$$J(v) = \langle (\Delta_g + h)^{-1}(J), v \rangle_h.$$

Since no ambiguity will occur, the notation $H^1(M)$ will also be used to denote the space of Sobolev 1-forms over M .

The blowing-up solutions of Theorem 1.1 are obtained by glueing degenerating peaks over the solution (u, T) . Here we define such peaks. Let $(r_k)_k$ be a sequence of positive real numbers which converges towards zero as $k \rightarrow \infty$ and satisfy:

$$\beta_k \ll r_k \ll d_g(\xi_k, \xi_{k+1}) \quad \text{and} \quad r_k^N \gg \mu_k, \quad (2.15)$$

where β_k is given by (2.6), and for some large enough integer N , as $k \rightarrow +\infty$. The lowest value of N required can be made explicit (only depending on n), as is easily seen in the proof of Theorem 1.1. Examples of sequences $(\tau_k)_k$, $(\beta_k)_k$ and $(r_k)_k$ that satisfy (2.6), (2.15) and $\sum_k \tau_k < +\infty$ are easily found, for instance as different powers of $(1/k)_k$. For $t > 0$ we define the sequence $(\delta_k(t))_k$ by:

$$\delta_k(t) = \mu_k t, \quad (2.16)$$

where μ_k is as in (2.5). We let $r_0 > 0$ be such that $r_0 < i_{g_\xi}(M)$ for all $\xi \in M$, where i_{g_ξ} denotes the injectivity radius of the metric g_ξ given by (2.2). By definition of r_k there holds $2r_k < r_0$ for any k . We let $\chi \in C^\infty(\mathbb{R})$ be a nonnegative, smooth compactly supported function such that $\chi \equiv 1$ in $[-1, 1]$ and $\chi \equiv 0$ outside of $[-2, 2]$. The blow-up profiles we investigate in this work are given by the following expression: for $t > 0$ and $\xi \in M$, and for any $x \in M$:

$$W_{k,t,\xi}(x) = \Lambda_\xi(x) \chi \left(\frac{d_{g_\xi}(\xi, x)}{r_k} \right) \delta_k^{\frac{n-2}{2}} \left(\delta_k^2 + \frac{f(\xi)}{n(n-2)} d_{g_\xi}(\xi, x)^2 \right)^{1-\frac{n}{2}}, \quad (2.17)$$

where Λ_ξ is as in (2.3) and δ_k is given by (2.16). These profiles are localized in $B_\xi(2r_k)$, which denotes here the geodesic ball of radius $2r_k$ with respect to the metric g_ξ . For a given $\xi \in M$ we let

$V_{0,\xi}, \dots, V_{n,\xi} : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by:

$$\begin{aligned} V_{0,\xi}(x) &= \left(\frac{f(\xi)}{n(n-2)} |x|^2 - 1 \right) \left(1 + \frac{f(\xi)}{n(n-2)} |x|^2 \right)^{-\frac{n}{2}} \\ V_{i,\xi}(y) &= f(\xi) x_i \left(1 + \frac{f(\xi)}{n(n-2)} |x|^2 \right)^{-\frac{n}{2}}. \end{aligned} \quad (2.18)$$

For any $0 \leq i \leq n$, the $V_{i,\xi}$ span the set of solutions in $H^1(\mathbb{R}^n)$ of the linearized equation (Bianchi-Egnell [5]):

$$\Delta_{\text{eucl}} V_{i,\xi} = (2^* - 1) f(\xi) U_\xi^{2^*-2} V_{i,\xi}, \quad (2.19)$$

where we have let:

$$U_\xi(y) = \left(1 + \frac{f(\xi)}{n(n-2)} |y|^2 \right)^{1-\frac{n}{2}}, \text{ for any } y \in \mathbb{R}^n. \quad (2.20)$$

We also define, for any $x \in M$, any $1 \leq i \leq n$ and any $\xi \in M$:

$$\begin{aligned} Z_{0,k,t,\xi}(x) &= \Lambda_\xi(x) \chi \left(\frac{d_{g_\xi}(\xi, x)}{r_k} \right) \delta_k^{\frac{n-2}{2}} \left(\delta_k^2 + \frac{f(\xi)}{n(n-2)} d_{g_\xi}(\xi, x)^2 \right)^{-\frac{n}{2}} \\ &\quad \times \left(\frac{f(\xi)}{n(n-2)} d_{g_\xi}(\xi, x)^2 - \delta_k^2 \right) \\ Z_{i,k,t,\xi}(x) &= \Lambda_\xi(x) \chi \left(\frac{d_{g_\xi}(\xi, x)}{r_k} \right) \delta_k^{\frac{n}{2}} \left(\delta_k^2 + \frac{f(\xi)}{n(n-2)} d_{g_\xi}(\xi, x)^2 \right)^{-\frac{n}{2}} \\ &\quad \times f(\xi) \left\langle \left(\exp_{\xi}^{g_\xi} \right)^{-1}(x), e_i(\xi) \right\rangle_{g_\xi(\xi)}. \end{aligned} \quad (2.21)$$

In (2.21), the $(e_i)_i$ denote the field of orthonormal basis centered at ξ_0 introduced in the beginning of this Section. Finally, we let

$$K_{k,t,\xi} = \text{Span} \{ Z_{i,k,t,\xi}, i = 0 \dots n \}. \quad (2.22)$$

Since $(V_{0,\xi}, \dots, V_{n,\xi})$ forms an orthonormal family for the scalar product $(u, v) = \int_{\mathbb{R}^n} \langle \nabla u, \nabla v \rangle dx$ in \mathbb{R}^n , $K_{k,t,\xi}$ is $(n+1)$ -dimensional for k large enough and the $Z_{i,k,t,\xi}$ are “almost” orthogonal. We denote by $K_{k,t,\xi}^\perp$ its orthogonal in $H^1(M)$ for the scalar product given by (2.14).

We did not specify the choice of f and X yet. Let $f_0 > 0$ be a positive constant. Here f will be a perturbation of f_0 by small bumps:

$$f = f_0 + \sum_{k \geq 0} s_k \chi \left(\frac{1}{r_k} (\exp_{\xi_k}^{g_{\xi_k}})^{-1}(x) \right), \quad (2.23)$$

where $(s_k)_k$ is a sequence of real numbers converging to zero and satisfying $|s_k| = O(\mu_k^N)$ for a sufficiently large $N \in \mathbb{N}^*$. Let X_0 denote any smooth field of 1-forms in M which vanishes in a neighbourhood of ξ_0 . Let Z be a fixed smooth 1-form in \mathbb{R}^n , compactly supported in $B_0(1)$, and with $|Z_0(0)| > 0$. Define then, for any $x \in M$:

$$X(x) = X_0(x) + \sum_{k \geq 0} \mu_k^{\frac{n-1}{2}} Z \left(\frac{1}{r_k} (\exp_{\xi_k}^{g_{\xi_k}})^{-1}(x) \right), \quad (2.24)$$

where μ_k and r_k are as in (2.5) and (2.15). Up to reducing $\|X_0\|_\infty$ and the τ_k such an X always satisfies (2.11). In the following, we will also let:

$$\alpha := |Z(0)| = \frac{|X(\xi_k)|_{g_{\xi_k}}}{\mu_k^{\frac{n-1}{2}}} \text{ for all } k. \quad (2.25)$$

This parameter α will have to be chosen small, see the proof of Proposition 6.1. Again, with (2.5) and (2.15), f and X can always be chosen to belong to $C^r(M)$ for $r \in \mathbb{N}^*$.

Finally, let

$$\mathcal{E} = \left\{ (\varepsilon_k)_{k \in \mathbb{N}}, \varepsilon_k > 0, \lim_{k \rightarrow \infty} \varepsilon_k = 0 \right\} \quad (2.26)$$

be the set of sequences of positive real numbers converging to 0. For $(\varepsilon_k)_k \in \mathcal{E}$ and for a given value of $(t, \xi) \in (0, +\infty) \times M$ we define the following sequence of subsets of $C^2(M)$:

$$F_k = F(\varepsilon_k, t, \xi) = \left\{ v \in C^0(M) \text{ such that } \left\| \frac{v}{u + W_{k,t,\xi}} \right\|_{C^0(M)} \leq \varepsilon_k \right\}, \quad (2.27)$$

where $u = u(X)$ is defined after (2.12) and $W_{k,t,\xi}$ is as in (2.17).

3. AN H^1 FINITE-DIMENSIONAL REDUCTION METHOD FOR THE SEMI-DECOUPLED SYSTEM

As discussed in the Introduction, the supercritical nonlinear coupling of system (1.1) makes the usual variational energy methods ineffective. In this Section we perform the first step of the fixed-point procedure in the proof of Theorem 1.1.

As a first step in view of the H^1 -theory, we get rid of the negative non-linearity in (1.1). We let, for any $\varepsilon > 0$ and $r \in \mathbb{R}$:

$$\rho_\varepsilon(r) = \begin{cases} \varepsilon & \text{if } r < \varepsilon \\ r & \text{if } r \geq \varepsilon. \end{cases}$$

We let T denote any field of 1-forms and we introduce the following truncation of the scalar equation of (1.1):

$$\Delta_g u + hu = fu^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|^2 + \pi^2}{\rho_\varepsilon(u)^{2^*+1}}. \quad (3.1)$$

The following Lemma, proven in Premoselli-Wei [40], holds:

Lemma 3.1. *There exists $\varepsilon_0 > 0$ depending only on g, h, f, π such that for any 1-form T and any $0 \leq \varepsilon \leq \varepsilon_0$, any C^2 positive solution u of (3.1) satisfies:*

$$\min_M u \geq \varepsilon_0.$$

In particular, for $0 \leq \varepsilon \leq \varepsilon_0$, any C^2 positive solution of (3.1) also solves:

$$\Delta_g u + hu = fu^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|^2 + \pi^2}{u^{2^*+1}}.$$

In the following, if ε_0 is given by Lemma 3.1, we will let:

$$\rho = \rho_\varepsilon \text{ for some fixed } 0 < \varepsilon \leq \frac{1}{4}\varepsilon_0. \quad (3.2)$$

We now decouple system (1.1). Let $(\varepsilon_k)_k \in \mathcal{E}$, $(t, \xi) \in (0, +\infty) \times M$ and $v_k \in F_k = F(\varepsilon_k, t, \xi)$, where \mathcal{E} and F_k are defined in (2.26) and (2.27). To every v_k we associate a unique 1-form $T_k = T_{k,t,\xi}$ defined as the unique solution in M of

$$\vec{\Delta}_g T_k = \left(u + W_{k,t,p} + v_k \right)^{2^*} X + Y. \quad (3.3)$$

Such a 1-form T_k is unique since (M, g) possesses no conformal Killing fields, and Proposition 9.2 provides us with sharp pointwise asymptotics on T_k . We introduce the following equation in M , which is the scalar equation of (1.1) with an additional source term, of unknown w :

$$\Delta_g w + hw = fw^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|^2 + \pi^2}{\rho(w)^{2^*+1}} + \frac{|\mathcal{L}_g T_{k,t,\xi} + \sigma|^2_g - |\mathcal{L}_g T + \sigma|^2_g}{(u + W_{k,t,\xi_k} + v_k)^{2^*+1}}, \quad (3.4)$$

where $T_{k,t,\xi}$ is given by (3.3) and (u, T) are as in (2.12). Choosing v_k and investigating equation (3.4) amounts to semi-decoupling the system (1.1). In this section we solve equation (3.4) *via* a standard H^1 finite-dimensional reduction.

Classically, the first step consists in showing that the linearized operator for the scalar equation of (1.1) can be inverted in the orthogonal of $K_{k,t,\xi}$:

Proposition 3.2. *Let $D > 0$. There exists a positive constant C such that for any $(t, \xi) \in [1/D, D] \times M$ and for any k there holds:*

$$\frac{1}{C} \|\psi\|_{H^1(M)} \leq \|L_{k,t,\xi}(\psi)\|_{H^1(M)} \leq C \|\psi\|_{H^1(M)} \text{ for any } \psi \in K_{k,t,\xi}, \quad (3.5)$$

where $K_{k,t,\xi}$ is as in (2.22) and where we have let, for any $\psi \in K_{k,t,\xi}$:

$$L_{k,t,\xi}(\psi) = \Pi_{K_{k,t,\xi}^\perp} \left(\psi - (\Delta_g + h)^{-1} \left[(2^* - 1) f(u + W_{k,t,\xi})^{2^*-2} \psi - (2^* + 1) \frac{|\mathcal{L}_g T + \sigma|^2 + \pi^2}{(u + W_{k,t,\xi})^{2^*+2}} \psi \right] \right), \quad (3.6)$$

where $\Pi_{K_{k,t,\xi}^\perp}$ denotes the orthogonal projection on $K_{k,t,\xi}^\perp$ with respect to the scalar product given in (2.14). In (3.6), $(u, T) = (u(X), T(X))$ are defined in the discussion following (2.12).

The proof of Proposition 3.2 follows from standard arguments and is clearly detailed in Robert-Vetois [45]. The main result of this section is the resolution of the full non-linear equation (3.4), given by the following Proposition:

Proposition 3.3. *Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k \gg \mu_k^{\frac{3}{2}}$ as $k \rightarrow +\infty$, where μ_k is defined in (2.5). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$ and, for any k , let $v_k \in F_k = F(\varepsilon_k, t_k, \xi_k)$. For k large enough, there exists a function $\phi_k = \phi_k(t_k, \xi_k, v_k) \in K_{k,t_k,\xi_k}^\perp$ that solves the following equation:*

$$\begin{aligned} & \Pi_{K_{k,t_k,\xi_k}^\perp} \left\{ u + W_{k,t_k,\xi_k} + \phi_k \right. \\ & - (\Delta_g + h)^{-1} \left(f(u + W_{k,t_k,\xi_k} + \phi_k)^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{\rho(u + W_{k,t_k,\xi_k} + \phi_k)^{2^*+1}} \right) \\ & \left. - (\Delta_g + h)^{-1} \left(\frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_{k,t_k,\xi_k} + v_k)^{2^*+1}} \right) \right\} = 0. \end{aligned} \quad (3.7)$$

This ϕ_k is the unique solution of (3.7) in $K_{k,t_k,\xi_k}^\perp \cap B_{H^1(M)}(0, C\eta\varepsilon_k)$, where C denotes some positive constant that does not depend on k , on the choice of $(t_k, \xi_k)_k$ or on η as in (2.11). Also, in (3.7), K_{k,t_k,ξ_k} is as in (2.22), ρ is as in (3.2), and T_k is as in (3.3).

As before, in (3.7) $(u, T) = (u(X), T(X))$ denote the specific solution of (2.12) obtained by the implicit function theorem when (2.11) holds. As an obvious consequence of Proposition 3.3, the function ϕ_k constructed therein satisfies:

$$\phi_k \in K_{k,t_k,\xi_k}^\perp \text{ for all } (t_k, \xi_k)_k \in [1/D, D] \times M, \quad (3.8)$$

and

$$\|\phi_k\|_{H^1(M)} \leq C\eta\varepsilon_k, \quad (3.9)$$

for some constant C which is independent of $(t_k, \xi_k)_k$, k and η .

Note that the necessity to introduce a source term in (3.4) is due to the blow-up behavior of $\mathcal{L}_g T_k$ as given by Proposition 9.2 below. Here, in the setting of Proposition 3.3, the finite-dimensional reduction will work since the nonlinearity $u \mapsto (|\mathcal{L}_g T + \sigma|^2 + \pi^2)\rho(u)^{-2^*+1}$ is of subcritical type in the sense of Robert-Vétois [45]. But, as can be easily checked, there is no hope to even get an analogue of Proposition 3.2 if instead we considered $(|\mathcal{L}_g T_k + \sigma|^2 + \pi^2)\rho(u)^{-2^*+1}$, with T_k given by (3.3).

Proof. Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$. Assume that

$$\varepsilon_k \gg \mu_k^{\frac{3}{2}} \quad (3.10)$$

where μ_k is as in (2.5). Let $(t_k, \xi_k)_k \in [1/D, D] \times M$ and $(v_k)_k, v_k \in F_k = F_{\varepsilon_k, t_k, \xi_k}$ be fixed. In the proof of Proposition 3.3, for the sake of clarity and since no ambiguity will occur, we will omit the dependence in t_k and ξ_k in the quantities appearing. Hence we shall denote K_{k, t_k, ξ_k} by K_k , W_{k, t_k, ξ_k} by W_k and so on. Let $\phi \in K_k^\perp$ be fixed. It is easily seen that ϕ solves:

$$\begin{aligned} \Pi_{K_k^\perp} \left\{ u + W_k + \phi \right. \\ \left. - (\Delta_g + h)^{-1} \left(f(u + W_k + \phi)^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{\rho(u + W_k + \phi)^{2^*+1}} \right) \right. \\ \left. + (\Delta_g + h)^{-1} \left(\frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}} \right) \right\} = 0 \end{aligned}$$

if and only if ϕ solves the fixed-point equation:

$$\phi = \Theta_k(\phi) \quad \text{in } K_k^\perp,$$

where we have let

$$\Theta_k(\phi) = L_k^{-1} \circ \Pi_{K_k^\perp} \circ (\Delta_g + h)^{-1} \circ N_k(\phi) - L_k^{-1} \circ \Pi_{K_k^\perp}(R_k), \quad (3.11)$$

where $L_k = L_{k, t_k, \xi_k}$ is as in (3.6) and where in (3.11) we have let

$$\begin{aligned} N_k(\phi) = f \left[(u + W_k + \phi)^{2^*-1} - (u + W_k)^{2^*-1} - (2^* - 1)(u + W_k)^{2^*-2} \phi \right] \\ + (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left[\rho(u + W_k + \phi)^{-2^*-1} - (u + W_k)^{-2^*-1} + (2^* + 1)(u + W_k)^{-2^*-2} \phi \right] \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} R_k = u + W_k - (\Delta_g + h)^{-1} \left(f(u + W_k)^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{(u + W_k)^{2^*+1}} \right) \\ + (\Delta_g + h)^{-1} \left(\frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}} \right). \end{aligned} \quad (3.13)$$

Note that Θ_k is well-defined because of Proposition 3.2. We now apply Banach-Picard's fixed-point theorem to Θ_k defined in (3.11). Let $\phi_1, \phi_2 \in K_k^\perp$. Standard computations show that there holds, for some positive constant C independent of k and η as in (2.11):

$$\begin{aligned} \left\| f \left[(u + W_k + \phi_1)^{2^*-1} - (u + W_k + \phi_2)^{2^*-1} - (2^* - 1)(u + W_k)^{2^*-2}(\phi_1 - \phi_2) \right] \right\|_{(H^1(M))'} \\ \leq C \left(\|\phi_1\|_{H^1(M)}^{\frac{4}{n-2}} + \|\phi_2\|_{H^1(M)}^{\frac{4}{n-2}} \right) \|\phi_1 - \phi_2\|_{H^1(M)}, \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} & \left\| \left(|\mathcal{L}_g T + \sigma|_g^2 + \pi^2 \right) \left[\rho(u + W_k + \phi_1)^{-2^*-1} - \rho(u + W_k + \phi_2)^{-2^*-1} \right. \right. \\ & \quad \left. \left. + (2^* + 1)(u + W_k)^{-2^*-2}(\phi_1 - \phi_2) \right] \right\|_{(H^1(M))'} \\ & \leq C \sup_{v \in [\phi_1, \phi_2]} \left\| \frac{\rho'(u + W_k + v)}{\rho(u + W_k + v)^{2^*+2}} - \frac{1}{(u + W_k)^{2^*+2}} \right\|_{L^{\frac{n}{2}}(M)} \|\phi_1 - \phi_2\|_{H^1(M)}. \end{aligned} \quad (3.15)$$

Using the definition of ρ in (3.2) one obtains that for any $v \in [\phi_1, \phi_2] \subset K_k^\perp$:

$$\left\| \frac{\rho'(u + W_k + v)}{\rho(u + W_k + v)^{2^*+2}} - \frac{1}{(u + W_k)^{2^*+2}} \right\|_{L^{\frac{n}{2}}(M)} \leq C \|v\|_{H^1(M)}^{\frac{4}{n-2}}, \quad (3.16)$$

where C is a positive constant that does not depend on k, η or $v \in [\phi_1, \phi_2]$. Gathering (3.14), (3.15) and (3.16) one therefore has that:

$$\|\Theta_k(\phi_1) - \Theta_k(\phi_2)\|_{H^1(M)} \leq C_1 \left(\|\phi_1\|_{H^1(M)}^{\frac{4}{n-2}} + \|\phi_2\|_{H^1(M)}^{\frac{4}{n-2}} \right) \|\phi_1 - \phi_2\|_{H^1(M)}, \quad (3.17)$$

for some positive constant C_1 that neither depends on k nor on η . We now estimate $\Theta_k(0)$. By Proposition 3.2 there holds:

$$\|\Theta_k(0)\|_{H^1(M)} \leq C \|(\Delta_g + h)R_k\|_{L^{\frac{2n}{n+2}}(M)}, \quad (3.18)$$

for some positive C independent of k and η , where R_k is defined in (3.13). Straightforward computations using (2.7), (2.10) and (2.23) show that:

$$\left\| (\Delta_g + h)(u + W_k) - f(u + W_k)^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{(u + W_k)^{2^*+1}} \right\|_{L^{\frac{2n}{n+2}}(M)} = o(\delta_k^{\frac{3}{2}}) \quad (3.19)$$

as $k \rightarrow +\infty$, and the $o(\delta_k^{\frac{3}{2}})$ term in (3.19) is uniform in the choice of $(t_k, \xi_k)_k \in [1/D, D] \times M$. We now write that

$$|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2 = \langle \mathcal{L}_g T_k - \mathcal{L}_g T, \mathcal{L}_g T_k + \mathcal{L}_g T + 2\sigma \rangle_g$$

where the scalar product $\langle \cdot, \cdot \rangle_g$ is the standard one induced by the metric g on $(2, 0)$ -tensors. Using the pointwise estimates on $|\mathcal{L}_g T_k - \mathcal{L}_g T|_g$ given by (9.11), (2.24) and (2.15) one finds that:

$$\left\| \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}} \right\|_{L^{\frac{2n}{n+2}}(M)} \leq C' \left(\eta \varepsilon_k + \delta_k^{\frac{n}{4}} \right), \quad (3.20)$$

for some positive C' independent of k and η . Remember that throughout this paper we always assume that $n \geq 6$, so (3.18), (3.19) and (3.20) show that there exists a positive constant C_2 independent of k and η such that:

$$\|\Theta_k(0)\|_{H^1(M)} \leq C_2 \left(\eta \varepsilon_k + \delta_k^{\frac{3}{2}} \right). \quad (3.21)$$

We let now

$$c_k = 2C_2 \eta \varepsilon_k, \quad (3.22)$$

where C_2 is given by (3.21) and η is as in (2.11). With (3.17), (3.21) and (3.10) we obtain that, for any $\phi_1, \phi_2 \in K_k^\perp \cap B_{H^1(M)}(0, c_k)$, there holds:

$$\|\Theta_k(\phi_1) - \Theta_k(\phi_2)\|_{H^1(M)} \leq C_1 \left(2c_k^{\frac{4}{n-2}} \right) \|\phi_1 - \phi_2\|_{H^1(M)} \quad (3.23)$$

and

$$\|\Theta_k(\phi_1)\|_{H^1(M)} \leq \left(\frac{1}{2} + o(1) \right) c_k. \quad (3.24)$$

Up to choosing k large enough, (3.23) and (3.24) show that Θ_k is $\frac{1}{2}$ -Lipschitz from $B_{H^1(M)}(0, 2c_k)$ into itself. Banach-Picard's fixed-point theorem applies and, with (3.21), provides us with a function $\phi_k \in K_k^\perp$ satisfying (3.7), (3.8), (3.9), and which is the only solution of (3.7) in $K_k \cap B_{H^1(M)}(0, c_k)$, where c_k is as in (3.22). \square

Let us point out again that the proof of Proposition 3.3 crucially relies on the *pointwise* estimates available on v_k , in particular for estimate (3.20).

4. POINTWISE C^0 ESTIMATES ON ϕ_k

Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$, defined in (2.26). Assume that $\varepsilon_k \gg \mu_k^{\frac{3}{2}}$ as $k \rightarrow +\infty$, where μ_k is defined in (2.5). Let $(t_k, \xi_k)_k$ be any sequence in $[1/D, D] \times M$. For any $v_k \in F_k = F((\varepsilon_k)_k, t_k, \xi_k)$, Proposition 3.3 shows the existence of a function $\phi_k = \phi_k(t_k, \xi_k, v_k) \in K_{k, t_k, \xi_k}^\perp$ that solves (3.7) and satisfies (3.8) and (3.9).

As discussed in the Introduction, the proof of Theorem 1.1 goes through the application of a Banach-Picard fixed-point theorem to the mapping $v_k \mapsto \phi_k$. This requires to show in particular that such a mapping leaves F_k defined in (2.27) invariant or, in other words, to obtain an explicit pointwise control on ϕ_k . This task is achieved in this section and in the two following ones.

From here until the end of the paper, if $(f_k)_k$ denotes some sequence of real numbers or some sequence of functions, the notation $O(f_k)$ will denote a quantity whose absolute value can be bounded by the product of $|f_k|$ and of a constant *independent of k and η as in (2.11)*. The notation $o(f_k)$ is defined accordingly. Similarly, we will write “ $f_k \lesssim g_k$ ” when there exists a positive constant C *independent of k and η as in (2.11)* such that $f_k \leq Cg_k$ for any k .

In this section we obtain a first, rough, asymptotic pointwise control of ϕ_k :

Proposition 4.1. *Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k \gg \mu_k^{\frac{3}{2}}$ as $k \rightarrow +\infty$, where μ_k is defined in (2.5). Let $(t_k, \xi_k)_k$ be a sequence of points in $[1/D, D] \times M$, and let $v_k \in F_k = F(\varepsilon_k, t_k, \xi_k)$. There exists a sequence $(\nu_k)_k$ of positive numbers that goes to zero as $k \rightarrow +\infty$ such that:*

$$|\phi_k(x)| \leq \nu_k \left(u(x) + W_{k, t_k, \xi_k}(x) \right) \quad \text{for any } x \in M. \quad (4.1)$$

In (4.1) we have let $\phi_k = \phi_k(t_k, \xi_k, v_k) \in K_{k, t_k, \xi_k}$ be the solution of (3.7) given by Proposition 3.3.

As a consequence of Proposition 4.1, Lemma 3.1 and Proposition 3.3 one has that for any $(t_k, \xi_k)_k \in [1/D, D] \times M$ and for any $v_k \in F(\varepsilon_k, t_k, \xi_k)$, letting

$$u_{k, t_k, \xi_k, v_k} = u + W_{k, t_k, \xi_k} + \phi_k(t_k, \xi_k, v_k), \quad (4.2)$$

there exist real numbers $(\lambda_k^i(t_k, \xi_k, v_k))_{0 \leq i \leq n}$ such that u_{k, t_k, ξ_k, v_k} satisfies:

$$\begin{aligned} (\Delta_g + h)u_{k, t_k, \xi_k, v_k} - fu_{k, t_k, \xi_k, v_k}^{2^*-1} - \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u_{k, t_k, \xi_k, v_k}^{2^*+1}} \\ - \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_{k, t_k, \xi_k} + v_k)^{2^*+1}} = \sum_{i=0}^n \lambda_k^i(t_k, \xi_k, v_k) (\Delta_g + h) Z_{i, k}. \end{aligned} \quad (4.3)$$

Here again, T_{k,t_k,ξ_k} is as in (3.3).

Proof. Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k \gg \mu_k^{\frac{3}{2}}$ as $k \rightarrow +\infty$, where μ_k is defined in (2.5). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, let $v_k \in F_k = F(\varepsilon_k, t_k, \xi_k)$ and let $\phi_k = \phi_k(t_k, \xi_k, v_k) \in K_{k,t_k,\xi_k}^\perp$ be the solution of (3.7) given by Proposition 3.3. In particular, for any k large enough, there exist $(\lambda_k^i)_{0 \leq i \leq n} = (\lambda_k^i(t_k, \xi_k, v_k))_{0 \leq i \leq n}$ such that ϕ_k solves:

$$\begin{aligned} (\Delta_g + h)(u + W_k + \phi_k) - f(u + W_k + \phi_k)^{2^*-1} - \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{\rho(u + W_k + \phi_k)^{2^*+1}} \\ - \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}} = \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k}, \end{aligned} \quad (4.4)$$

where the $Z_{i,k}$ are defined in (2.21) and where T_k is as in (3.3). As before, in (4.4) and later on we shall omit the dependence in t_k and ξ_k and let $W_k = W_{k,t_k,\xi_k}$ and so on. A first, obvious remark is that (4.4) and standard elliptic regularity results show that $u + W_k + \phi_k$ belongs to $C^1(M)$, and then so does ϕ_k . The proof of Proposition 4.1 goes through several steps.

Step 1: Estimation of the λ_k^i . We show that there holds:

$$\sum_{i=0}^n |\lambda_k^i| = O(\eta \varepsilon_k) + O(\delta_k^{\frac{3}{2}}). \quad (4.5)$$

For this, we rewrite (4.4) as:

$$\begin{aligned} (\Delta_g + h)R_k + (\Delta_g + h)\phi_k - f\left[(u + W_k + \phi_k)^{2^*-1} - (u + W_k)^{2^*-1}\right] \\ - (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(\rho(u + W_k + \phi_k)^{-2^*-1} - (u + W_k)^{-2^*-1}\right) \\ = \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k}, \end{aligned} \quad (4.6)$$

where R_k is as in (3.13). Let $0 \leq i \leq n$ be fixed. There holds, by (3.9), (3.19), (3.20) and Hölder's inequality that:

$$\langle R_k + \phi_k, Z_{i,k} \rangle_h = O(\delta_k^{\frac{3}{2}}) + O(\eta \varepsilon_k). \quad (4.7)$$

Since we have:

$$\left| (u + W_k + \phi_k)^{2^*-1} - (u + W_k)^{2^*-1} \right| = O\left((u + W_k)^{2^*-2} |\phi_k|\right) + O\left(|\phi_k|^{2^*-1}\right),$$

there holds that

$$\int_M f\left[(u + W_k + \phi_k)^{2^*-1} - (u + W_k)^{2^*-1}\right] Z_{i,k} dv_g = O(\eta \varepsilon_k). \quad (4.8)$$

The sequence of functions $\left((|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(\rho(u + W_k + \phi_k)^{-2^*-1} - (u + W_k)^{-2^*-1}\right)\right)_k$ is uniformly bounded in $L^\infty(M)$, so we have, since $n \geq 6$:

$$\int_M (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(\rho(u + W_k + \phi_k)^{-2^*-1} - (u + W_k)^{-2^*-1}\right) Z_{i,k} dv_g = o(\delta_k^{\frac{3}{2}}). \quad (4.9)$$

Finally, there holds for any $0 \leq j \leq n$, that:

$$\langle Z_{i,k}, Z_{j,k} \rangle_h = \delta_{ij} \|\nabla V_{i,\xi}\|_{L^2(\mathbb{R}^n)}^2 + o(1), \quad (4.10)$$

where $V_{i,\xi}$ is as in (2.18). Multiplying (4.6) by $Z_{i,k}$, integrating and using (4.7) – (4.10) yields (4.5).

Step 2: Local behavior of ϕ_k . In this step we show that:

$$\delta_k^{\frac{n-2}{2}} u_k \left(\exp_{\xi_k}^{g_{\xi_k}}(\delta_k \cdot) \right) \longrightarrow U_{\xi_0} \text{ in } C_{loc}^1(\mathbb{R}^n), \quad (4.11)$$

as $k \rightarrow +\infty$, where $\xi_0 = \lim_{k \rightarrow +\infty} \xi_k$, U_{ξ_0} is defined in (2.20), δ_k in (2.16) and the exponential map for g_{ξ_k} is as in (2.1) and (2.2). In order to prove (4.11), we define \tilde{u}_k in $B_0(r_k/\delta_k)$ by:

$$\tilde{u}_k(x) = \delta_k^{\frac{n-2}{2}} u_k \left(\exp_{\xi_k}^{g_{\xi_k}}(\delta_k x) \right). \quad (4.12)$$

Remember that r_k is a positive radius given by (2.15) and that W_k is supported in $B_{\xi_k}(2r_k)$, the ball being taken for the metric g_{ξ_k} . It is easily seen that for any $x \in B_0(r_k/\delta_k)$, \tilde{u}_k satisfies:

$$\begin{aligned} \Delta_{g_k} \tilde{u}_k(x) + \delta_k^2 h(x_k) \tilde{u}_k(x) &= f(x_k) \tilde{u}_k(x)^{2^*-1} + \delta_k^{\frac{n+2}{2}} \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{\rho(u + W_k + \phi_k)^{2^*+1}}(x_k) \\ &+ \delta_k^{\frac{n+2}{2}} \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}}(x_k) + \sum_{i=0}^n \lambda_k^i \delta_k^{\frac{n+2}{2}} (\Delta_g + h) Z_{i,k}(x_k), \end{aligned} \quad (4.13)$$

where, in (4.13), we have let $x_k = \exp_{\xi_k}^{g_{\xi_k}}(\delta_k x)$ and $g_k = \left(\exp_{\xi_k}^{g_{\xi_k}} \right)^* g_{\xi_k}(\delta_k \cdot)$. By definition of ρ in (3.2) there holds:

$$\delta_k^{\frac{n+2}{2}} \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{\rho(u + W_k + \phi_k)^{2^*+1}}(x_k) \longrightarrow 0 \text{ in } C_{loc}^0(\mathbb{R}^n). \quad (4.14)$$

Using (9.11) below, (2.24), (2.15) and the pointwise control on v_k given by the definition of F_k in (2.27) we also have:

$$\delta_k^{\frac{n+2}{2}} \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}}(x_k) \longrightarrow 0 \text{ in } C_{loc}^0(\mathbb{R}^n). \quad (4.15)$$

The Laplacian of $Z_{i,k}$ is computed using (9.2) below, and with (4.5) we also obtain that:

$$\sum_{i=0}^n \lambda_k^i \delta_k^{\frac{n+2}{2}} (\Delta_g + h) Z_{i,k}(x_k) \longrightarrow 0 \text{ in } L_{loc}^\infty(\mathbb{R}^n). \quad (4.16)$$

By definition of u_k in (4.2) and \tilde{u}_k in (4.12) there holds, for any $x \in \mathbb{R}^n$, that:

$$\lim_{r \rightarrow 0} \limsup_{k \rightarrow +\infty} \int_{B_x(r)} \tilde{u}_k^{2^*} dv_{g_k} = 0,$$

and therefore an adaptation of Trudinger's standard argument (see for instance Hebey [25], theorem 2.15) along with (4.13), (4.14), (4.15) and (4.16) shows that \tilde{u}_k converges strongly in $C_{loc}^1(\mathbb{R}^n)$. By (3.9) and since $\varepsilon_k \rightarrow 0$, there holds

$$\int_K \delta_k^n \left| \phi_k \left(\exp_{\xi_k}^{g_{\xi_k}}(\delta_k x) \right) \right|^{2^*} dv_{g_k} = \int_{\exp_{\xi_k}^{g_{\xi_k}}(\delta_k K)} |\phi_k|^{2^*} dv_g = o(1),$$

so that using the explicit expression (4.2) of u_k , we see that (4.11) holds true.

Step 3: A lower-bound on ϕ_k . We now show that for any sequence $(x_k)_k$ of points of M there holds:

$$\phi_k(x_k) \geq o(u(x_k)) + o(W_k(x_k)). \quad (4.17)$$

Let G be the Green's function of the operator $\Delta_g + h$ in M and let $(x_k)_k$ be a sequence of points in M . First, by the definition of ρ in (3.2), since $\varepsilon_k \rightarrow 0$, by (3.9) and Fatou's lemma, we have:

$$\begin{aligned} & \int_M \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{\rho(u + W_k + \phi_k)^{2^*+1}}(y) G(x_k, y) dv_g(y) \\ & \geq \int_M \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}}(y) G(x_k, y) dv_g(y) + o(1) \end{aligned} \quad (4.18)$$

as $k \rightarrow +\infty$. Let $(A_k)_k$ denote some sequence of real numbers such that $A_k \rightarrow +\infty$ as $k \rightarrow \infty$ and such that $A_k \delta_k \rightarrow 0$ as $k \rightarrow \infty$. By (2.17) and (3.9), $W_k + \phi_k \rightarrow 0$ almost everywhere in $M \setminus B_{x_k}(A_k \delta_k)$ and thus Fatou's lemma gives again that:

$$\begin{aligned} & \int_{M \setminus B_{x_k}(A_k \delta_k)} f(u + W_k + \phi_k)^{2^*-1}(y) G(x_k, y) dv_g(y) \\ & \geq \int_M f u^{2^*-1}(y) G(x_k, y) dv_g(y) + o(1). \end{aligned} \quad (4.19)$$

Using (2.17), (3.9), the local convergence given in (4.11) and standard properties of Green's function (see e.g. Robert [43]), Fatou's lemma also shows that:

$$\int_{B_{x_k}(A_k \delta_k)} f(u + W_k + \phi_k)^{2^*-1}(y) G(x_k, y) dv_g(y) \geq (1 + o(1)) W_k(x_k) + o(1) \quad (4.20)$$

as $k \rightarrow +\infty$ (see for instance Hebey [25], proposition 6.1). Independently, using the pointwise estimate given by (9.11) below, the definition of X in (2.24), (2.15), the fact that $\varepsilon_k \rightarrow 0$ and standard properties of Green's function, one obtains that:

$$\int_M \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}}(y) G(x_k, y) dv_g(y) = o(1) \quad (4.21)$$

as $k \rightarrow +\infty$. Having in mind that $|Z_{i,k}| \lesssim W_k$ for any $0 \leq i \leq n$, it remains to write a Green's representation formula for the operator $\Delta_g + h$ in M and to use (4.4), (4.5), (4.18), (4.19), (4.20) and (4.21) to obtain that:

$$\begin{aligned} u_k(x_k) & \geq \int_M f u^{2^*-1}(y) G(x_k, y) dv_g(y) + \int_M \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}}(y) G(x_k, y) dv_g(y) \\ & \quad + (1 + o(1)) W_k(x_k) + o(1). \end{aligned} \quad (4.22)$$

Since u solves the scalar equation of (2.12), a Green's representation formula for u with (4.2) and (4.22) then concludes the proof of (4.17).

Note that, by the definition of ρ in (3.2), (4.17) shows in particular that u_k in (4.2) actually satisfies:

$$\begin{aligned} & (\Delta_g + h)u_k - f u_k^{2^*-1} - \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u_k^{2^*+1}} \\ & - \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}} = \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k}. \end{aligned} \quad (4.23)$$

Step 4: A global weak estimate on ϕ_k . We prove now that

$$\theta_k(x)^{\frac{n-2}{2}} |\phi_k(x)| \longrightarrow 0 \text{ in } L^\infty(M) \quad (4.24)$$

as $k \rightarrow +\infty$, where we have let, for any $x \in M$:

$$\theta_k(x) = \delta_k + d_{g_{\varepsilon_k}}(\xi_k, x). \quad (4.25)$$

The proof of (4.24) proceeds by contradiction: assume that there exists a sequence $(x_k)_k$ in M such that

$$\theta_k(x_k)^2 |\phi_k(x_k)|^{2^*-2} = \max_{x \in M} \theta_k(x)^2 |\phi_k(x)|^{2^*-2} \geq \varepsilon_0 \quad (4.26)$$

for some $\varepsilon_0 > 0$. We start by noticing that there holds:

$$\theta_k(x_k)^2 W_k(x_k)^{2^*-2} = o(1) \quad (4.27)$$

and

$$u_k(x_k) \rightarrow +\infty \quad (4.28)$$

as $k \rightarrow +\infty$, where x_k is defined in (4.24). Equations (4.27) and (4.28) follow from a straightforward adaptation of the arguments in Hebey [25] (proposition 7.1) combined with (4.5), (4.11), (4.17), (9.11) and (4.23). We let in what follows

$$\tilde{\delta}_k = u_k(x_k)^{-\frac{2}{n-2}}.$$

Equation (4.28) implies in particular that $\tilde{\delta}_k \rightarrow 0$ as $k \rightarrow +\infty$. For any $x \in B_0(i_g(M)/\tilde{\delta}_k)$, we let:

$$\tilde{w}_k(x) = \tilde{\delta}_k^{\frac{n-2}{2}} u_k \left(\exp_{x_k}^{g_{x_k}}(\tilde{\delta}_k x) \right). \quad (4.29)$$

In case $\delta_k = o(\tilde{\delta}_k)$ and $d_{g_{x_k}}(x_k, \xi_k) = O(\tilde{\delta}_k)$, let $\mathcal{S} = \{\tilde{\xi}_0\}$, where $\tilde{\xi}_0 = \lim_{k \rightarrow +\infty} \frac{1}{\tilde{\delta}_k} \left(\exp_{x_k}^{g_{x_k}} \right)^{-1}(\xi_k)$. Otherwise, let $\mathcal{S} = \emptyset$. Let $K \subset \subset \mathbb{R}^n \setminus \mathcal{S}$ be a compact set. For any $z \in K$, let $z_k = \exp_{x_k}^{g_{x_k}}(\tilde{\delta}_k z)$. There holds in particular that

$$d_{g_{\xi_k}}(\xi_k, z_k) \geq C_0 \tilde{\delta}_k \quad (4.30)$$

for some positive constant C_0 . Using (4.26), (4.27) and (4.29) one easily obtains that for any $x \in K$,

$$\left| \tilde{w}_k(x) - \tilde{\delta}_k^{\frac{n-2}{2}} W_k(z_k) \right|^{2^*-2} = O(1), \quad (4.31)$$

and the constant in the $O(1)$ term obviously depends on K . We now claim that

$$\tilde{\delta}_k^{\frac{n-2}{2}} W_k(z_k) = o(1). \quad (4.32)$$

By (2.17) and (4.30), the only case where (4.32) is not clearly satisfied is when $\frac{1}{C} \delta_k \leq \tilde{\delta}_k \leq C \delta_k$ and $d_{g_{\xi_k}}(\xi_k, z_k) \leq C \delta_k$ for some positive constant C . In this case there also holds that $d_{g_{\xi_k}}(\xi_k, x_k) = O(\delta_k)$, hence $\liminf_{k \rightarrow +\infty} \tilde{\delta}_k^{\frac{n-2}{2}} W_k(x_k) > 0$ by definition of W_k , which contradicts (4.27). Hence (4.32) holds true. Coming back to (4.31) with (4.32) we have in particular:

$$\tilde{w}_k(x) \leq C_K \text{ for all } x \in K. \quad (4.33)$$

By construction of \tilde{w}_k in (4.29) there holds $\tilde{w}_k(0) = 1$. As an easy consequence of (4.26) and of the definition of \mathcal{S} above one always has $|\tilde{\xi}_0| > 0$ whenever $\tilde{\xi}_0$ is finite. Hence (4.32) can be applied to some compact subset $K \ni 0$ and yields

$$\tilde{\delta}_k^{\frac{n-2}{2}} W_k(x_k) = o(1) \quad (4.34)$$

as $k \rightarrow +\infty$ which, combined with (4.26), also gives:

$$\theta_k(x_k) \geq \frac{1}{C} \tilde{\delta}_k \quad (4.35)$$

for some positive C independent of k , where θ_k is as in (4.25). Let now $0 \in K \subset \subset \mathbb{R}^n \setminus \mathcal{S}$. By (4.23) and (4.29), \tilde{w}_k satisfies, for any $x \in K$:

$$\begin{aligned} \Delta_{\tilde{g}_k} \tilde{w}_k(x) + \tilde{\delta}_k^2 h(z_k) \tilde{w}_k(x) &= f(z_k) \tilde{w}_k(x)^{2^*-1} + \tilde{\delta}_k^{2n} \frac{(|\mathcal{L}_g T + \sigma|_g^2 + \pi^2)(z_k)}{\tilde{w}_k^{2^*+1}(x)} \\ &+ \tilde{\delta}_k^{\frac{n+2}{2}} \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}}(z_k) + \tilde{\delta}_k^{\frac{n+2}{2}} \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k}(z_k), \end{aligned} \quad (4.36)$$

where again $z_k = \exp_{x_k}^{g_{x_k}}(\tilde{\delta}_k x)$ and where $\tilde{g}_k = \left(\exp_{x_k}^{g_{x_k}}\right)^* g(\tilde{\delta}_k \cdot)$. Using estimate (9.11) below, (2.24) and using the pointwise estimates on v_k given by the choice $v_k \in F_k = F(\varepsilon_k, t_k, \xi_k)$ it is easily seen that there holds

$$\tilde{\delta}_k^{\frac{n+2}{2}} \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}}(z_k) = O\left(\tilde{\delta}_k^{\frac{n+2}{2}}\right) \text{ for any } x \in K. \quad (4.37)$$

Combining (4.32) with the pointwise expression of $(\Delta_h + h)Z_{i,k}$ given by (9.2) below, the inequality $|Z_{i,k}| \lesssim W_k$ for $0 \leq i \leq n$ and with (2.15) yields:

$$\tilde{\delta}_k^{\frac{n+2}{2}} \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k}(z_k) = o(1) \text{ uniformly in } K. \quad (4.38)$$

In the end, (4.33), (4.37) and (4.38) give with (4.36) that \tilde{w}_k satisfies in K :

$$\Delta_{\tilde{g}_k} \tilde{w}_k(x) + \tilde{\delta}_k^2 h(z_k) \tilde{w}_k(x) = f(z_k) \tilde{w}_k(x)^{2^*-1} + \frac{o(1)}{\tilde{w}_k^{2^*+1}}, \quad (4.39)$$

where the term $o(1)$ is uniform in K . With (4.33) and (4.39), the Harnack inequality for the Einstein-Lichnerowicz equation stated in Premoselli [39] (Proposition 6.1) then shows that there exists a positive constant C_K such that

$$\frac{1}{C_K} \leq \tilde{w}_k \leq C_K \text{ in } K \quad (4.40)$$

and thus, by standard elliptic theory, that \tilde{w}_k converges in $C_{loc}^2(\mathbb{R}^n \setminus \mathcal{S})$ towards a positive solution \tilde{w}_0 of

$$\Delta_\xi \tilde{w}_0 = f(x_0) \tilde{w}_0^{2^*-1} \text{ in } \mathbb{R}^n \setminus \mathcal{S},$$

where $x_0 = \lim_{k \rightarrow +\infty} x_k$. In particular, (4.40) shows that

$$\int_K \tilde{w}_0^{2^*} dy > 0 \text{ for any compact set } K \subset \subset \mathbb{R}^n \setminus \mathcal{S}. \quad (4.41)$$

But independently, by (4.29), (4.2), (4.32) and (3.9), one has that $\tilde{w}_k \rightarrow 0$ in $L^{2^*}(K)$ as $k \rightarrow +\infty$, which is a contradiction with (4.41). This concludes the proof of (4.24).

In the following we let, for any $\delta > 0$:

$$\nu_k(\delta) = \sup_{M \setminus B_{\xi_k}(\delta)} u_k. \quad (4.42)$$

Step 5: A first set of strong pointwise estimates. We show that for any $\varepsilon > 0$, there exist $R_\varepsilon > 0$, $\delta_\varepsilon > 0$ and $C_\varepsilon > 0$ such that:

$$u_k(x) \leq C_\varepsilon \left(\delta_k^{\frac{n-2}{2}(1-2\varepsilon)} d_{g_{\xi_k}}(\xi_k, x)^{(2-n)(1-\varepsilon)} + \nu_k(\delta_\varepsilon) d_{g_{\xi_k}}(\xi_k, x)^{(2-n)\varepsilon} \right), \quad (4.43)$$

for any k and for any $x \in M \setminus B_{\xi_k}(R_\varepsilon \delta_k)$. For $\varepsilon > 0$, we define the following function in M :

$$\Psi_{k,\varepsilon}(x) = \delta_k^{\frac{n-2}{2}(1-2\varepsilon)} G(\xi_k, x)^{1-\varepsilon} + \nu_k(\delta_\varepsilon) G(\xi_k, x)^\varepsilon, \quad (4.44)$$

where G is the Green's function of $\Delta_g + h$ in M . We let $R > 0$ and let $(x_k)_k$ be defined by:

$$\frac{u_k}{\Psi_{k,\varepsilon}}(x_k) = \sup_{M \setminus B_{\xi_k}(R\delta_k)} \frac{u_k}{\Psi_{k,\varepsilon}}. \quad (4.45)$$

We now claim that, up to choosing R large enough and δ small enough, there holds, for $k \gg 1$, that:

$$x_k \in \partial B_{\xi_k}(R\delta_k) \text{ or } d_{g_{\xi_k}}(\xi_k, x_k) \geq \delta. \quad (4.46)$$

The proof of (4.46) proceeds by contradiction: if we assume that (4.46) does not hold, then (4.45) gives that:

$$d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{\Delta_g \Psi_{k,\varepsilon}}{\Psi_{k,\varepsilon}}(x_k) \leq d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{\Delta_g u_k}{u_k}(x_k). \quad (4.47)$$

Straightforward computations show that

$$d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{\Delta_g \Psi_{k,\varepsilon}}{\Psi_{k,\varepsilon}}(x_k) \geq -(1-\varepsilon)(1-C\varepsilon)d_{g_{\xi_k}}(\xi_k, x_k)^2 + C\varepsilon(1-\varepsilon) \quad (4.48)$$

for some positive constant C independent of k . Independently, using (4.23) we have:

$$\begin{aligned} d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{\Delta_g u_k}{u_k}(x_k) &= -d_{g_{\xi_k}}(\xi_k, x_k)^2 h(x_k) + d_{g_{\xi_k}}(\xi_k, x_k)^2 f(x_k) u_k^{2^*-2}(x_k) \\ &+ d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u_k^{2^*+2}}(x_k) + d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{u_k(u + W_k + v_k)^{2^*+1}}(x_k) \\ &+ d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{1}{u_k} \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k}(x_k). \end{aligned} \quad (4.49)$$

By (4.2), (4.24) and since we assumed that (4.46) does not hold, we have that

$$d_{g_{\xi_k}}(\xi_k, x_k)^2 f(x_k) u_k^{2^*-2}(x_k) \leq o(1) + \|u\|_{L^\infty(M)}^{2^*-2} \delta^2 + C \left(1 + \frac{f(\xi_k)}{n(n-2)} R^2\right)^{-2} \quad (4.50)$$

for some positive constant C independent of k . Similarly, using in addition (4.17):

$$d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u_k^{2^*+2}}(x_k) \leq C' \delta^2 \quad (4.51)$$

for some positive C' independent of k . Here, (u, T) are always as in (2.12). Using the pointwise estimates on v_k given by (2.27), (2.24), (4.17), (2.15) and (9.11) below there holds:

$$\begin{aligned} d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{u_k(u + W_k + v_k)^{2^*+1}} \\ \leq \begin{cases} O(\delta_k) & \text{if } d_{g_{\xi_k}}(\xi_k, x_k) \leq \sqrt{\delta_k} \\ C(n, g, u_0) \delta^2 + o(1) & \text{if } d_{g_{\xi_k}}(\xi_k, x_k) \geq \sqrt{\delta_k}. \end{cases} \end{aligned} \quad (4.52)$$

Finally, since $|Z_{i,k}| \lesssim W_k$, by (9.2) and (2.15) one gets, mimicking the proof of (4.50), that:

$$\begin{aligned} d_{g_{\xi_k}}(\xi_k, x_k)^2 \frac{1}{u_k} \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k}(x_k) \\ \leq C(n, g, u_0, h) \delta^2 + C \left(1 + \frac{f(\xi_k)}{n(n-2)} R^2\right)^{-2} + o(1). \end{aligned} \quad (4.53)$$

For a fixed value of ε , (4.47), (4.50), (4.51), (4.52) and (4.53) give a contradiction with (4.48) up to choosing R large enough and δ small enough. This shows that (4.46) holds. Then (4.43) follows from (4.44) and the local convergence given by (4.11).

Step 6: End of the proof of (4.1). We show that there exists a sequence ν_k of positive real numbers converging to 0 as $k \rightarrow +\infty$ such that for any $x \in M$ there holds:

$$|\phi_k(x)| \leq \nu_k(W_k(x) + u(x)). \quad (4.54)$$

Let $(x_k)_k$ be a sequence of points in M . We prove in what follows that there holds:

$$|\phi_k(x_k)| = o(W_k(x_k)) + o(1). \quad (4.55)$$

Since $u > 0$ in M , (4.54) follows from (4.55). Assume first that $d_{g_{\xi_k}}(\xi_k, x_k) = O(\delta_k)$. Then (4.55) follows from (4.11). Assume then that $d_{g_{\xi_k}}(\xi_k, x_k) \not\rightarrow 0$ as $k \rightarrow +\infty$. Then (4.55) follows from (4.24). We may therefore assume that

$$\delta_k \ll d_{g_{\xi_k}}(\xi_k, x_k) \ll 1. \quad (4.56)$$

Let G denote again the Green's function of $\Delta_g + h$. By (4.5) and since $|Z_{i,k}(x_k)| \lesssim W_k(x_k)$ for $0 \leq i \leq n$ we easily get that

$$\int_M G(x_k, y) \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k}(y) dv_g(y) = \sum_{i=0}^n \lambda_k^i Z_{i,k}(x_k) = o(W_k(x_k)). \quad (4.57)$$

Independently, since $\left((|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) u_k^{-2^*-1} \right)_k$ is uniformly bounded in $L^\infty(M)$ by (4.17), we have that:

$$\int_M G(x_k, y) \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u_k^{2^*+1}}(y) dv_g(y) = \int_M G(x_k, y) \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}}(y) dv_g(y) + o(1) \quad (4.58)$$

as $k \rightarrow +\infty$. Now for some fixed $0 < \varepsilon < \frac{2}{n+2}$ we write that

$$\begin{aligned} & \int_M G(x_k, y) f(y) u_k^{2^*-1}(y) dv_g(y) \\ &= \int_{B_{\xi_k}(R_\varepsilon \delta_k)} G(x_k, y) f(y) u_k^{2^*-1}(y) dv_g(y) + \int_{M \setminus B_{\xi_k}(R_\varepsilon \delta_k)} G(x_k, y) f(y) u_k^{2^*-1}(y) dv_g(y). \end{aligned}$$

On the one side, (4.11) along with (4.56) shows that

$$\int_{B_{\xi_k}(R_\varepsilon \delta_k)} G(x_k, y) f(y) u_k^{2^*-1}(y) dv_g(y) = O(W_k(x_k)) + o(1). \quad (4.59)$$

On the other side, using (4.43) we obtain that:

$$\int_{M \setminus B_{\xi_k}(R_\varepsilon \delta_k)} G(x_k, y) f(y) u_k^{2^*-1}(y) dv_g(y) = O(W_k(x_k)) + O(1) \quad (4.60)$$

as $k \rightarrow +\infty$. Gathering (4.21), (4.57), (4.58), (4.59) and (4.60) and writing a representation formula for u_k gives, with (4.23), that :

$$|\phi_k(x_k)| \leq C(W_k(x_k) + u(x_k)), \quad (4.61)$$

for some positive constant C that neither depends on k nor on η in (2.11). In particular, C in (4.61) does not depend on the choice of $(t_k, \xi_k)_k$ and $(v_k)_k$. It remains to improve (4.61) into (4.55). Using the expression of the conformal laplacian of W_k given by (9.1) below, and since $f \in C^1(M)$, the following representation formula holds true for W_k :

$$W_k(x_k) = \int_M G(x_k, y) f(\xi_k) W_k^{2^*-1}(y) dv_g(y) + o(W_k(x_k)) + o(1), \quad (4.62)$$

where to obtain (4.62) we used that there holds, by (4.56) and Giraud's lemma (see [25], lemma 7.5):

$$\int_M G(x_k, y) W_k(y) dv_g(y) = O\left(\delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{4-n}\right) = O(\delta_k W_k(x_k)) + O(\delta_k), \quad (4.63)$$

where θ_k is defined in (4.25). We now write a representation formula for $u_k - W_k - u$, where u_k is defined in (4.2). Since u solves (2.12), by (4.23), (4.57), (4.58), (4.21) and (4.62) there holds:

$$\phi_k(x_k) = \int_M G(x_k, y) f(y) \left(u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right) (y) dv_g(y) + o(W_k(x_k)) + o(1). \quad (4.64)$$

By (4.11) there exists a sequence $A_k \rightarrow +\infty$ such that

$$\left\| \delta_k^{\frac{n-2}{2}} u_k \left(\exp_{\xi_k}^{g_{\xi_k}}(\delta_k \cdot) \right) - U_{\xi_0} \right\|_{C^0(B_0(A_k))} \rightarrow 0$$

as $k \rightarrow +\infty$, where U_{ξ_0} is defined in (2.20). We can always choose such a sequence $(A_k)_k$ to have $A_k \delta_k = o(\sqrt{\delta_k})$, so that there holds, by the dominated convergence theorem:

$$\int_{B_{\xi_k}(A_k \delta_k)} G(x_k, y) f(y) \left(u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right) (y) dv_g(y) = o(W_k(x_k)) + o(1). \quad (4.65)$$

By (2.17) and (4.61) there holds that

$$\left| u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right| \lesssim W_k + |\phi_k| \text{ in } M \setminus B_{\xi_k}(\sqrt{\delta_k}).$$

Since by (4.61) ϕ_k is uniformly bounded in $M \setminus B_{\xi_k}(\sqrt{\delta_k})$ we obtain with (3.9), (4.63) and since $\varepsilon_k = o(1)$ that

$$\int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} G(x_k, y) f(y) \left(u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right) (y) dv_g(y) = o(1) + o(W_k(x_k)). \quad (4.66)$$

By (4.61) we can write that there holds in $B_{\xi_k}(\sqrt{\delta_k})$:

$$\left| u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right| \lesssim W_k^{2^*-1}.$$

Since $A_k \rightarrow +\infty$ as $k \rightarrow +\infty$ straightforward computations therefore show that:

$$\int_{B_{\xi_k}(\sqrt{\delta_k}) \setminus B_{\xi_k}(A_k \delta_k)} G(x_k, y) f(y) \left(u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right) (y) dv_g(y) = o(W_k(x_k)) + o(1). \quad (4.67)$$

Equation (4.55) then follows from (4.64), (4.65), (4.66) and (4.67). \square

As a consequence of Proposition 4.1 we also obtain pointwise bounds on the gradient of ϕ_k . More precisely there holds:

$$|\nabla \phi_k(x)| \lesssim \left(1 + \delta_k^{\frac{n-2}{2}} \theta_k(x)^{1-n} \right), \quad (4.68)$$

where θ_k is defined in (4.25). Indeed, using Proposition 4.1, (2.24) and (9.11), equation (4.23) can be written as:

$$\triangle_g \phi_k + h \phi_k = O(W_k^{2^*-1}) + O(1),$$

so that writing again a representation formula for ϕ_k and differentiating yields easily (4.68).

5. SECOND-ORDER ESTIMATES ON ϕ_k .

Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$. Assume that $\varepsilon_k \gg \mu_k^{\frac{3}{2}}$ as $k \rightarrow +\infty$, where μ_k is defined in (2.5). Let $(t_k, \xi_k)_k \in [1/D, D] \times M$, for any k let $v_k \in F_k = F((\varepsilon_k)_k, t_k, \xi_k)$, and let $\phi_k = \phi_k(t_k, \xi_k, v_k)$ be given by Proposition 3.3.

The asymptotic pointwise estimates on ϕ_k obtained in Proposition 4.1 are not accurate enough to even say that ϕ_k belongs to $F(\varepsilon_k, t, \xi)$. In view of the final fixed-point argument we need to control ν_k introduced in (4.1) only in terms of ε_k and uniformly in the choice of $(t_k, \xi_k, v_k)_k$. This task is achieved in this section and in the following one.

In this Section we perform a second-order pointwise expansion of u_k defined in (4.2) and obtain finer pointwise estimates on ϕ_k .

We keep using the notations $f_k = O(g_k)$, $f_k = o(g_k)$ and $f_k \lesssim g_k$ introduced in Section 4. For the sake of clarity, we will also use the following notational shorthand: if $f \in L^\infty(M)$, $\|f\|_{L^\infty(2r_k)}$ will be used to denote the quantity $\|f\|_{L^\infty(B_{\xi_k}(2r_k))}$. Also, the notation $\mathbb{1}_{nlocf}$ will be used to denote a term which only appears when the manifold M is non-locally conformally flat.

We first obtain refined estimates on ϕ_k when the bubble W_k is the dominant term in the C^0 -decomposition of u_k :

Proposition 5.1. *Let $D > 0$, $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k \gg \mu_k^{\frac{3}{2}}$ as $k \rightarrow +\infty$, where μ_k is defined in (2.5). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, let $v_k \in F_k = F(\varepsilon_k, t_k, \xi_k)$ and let $\phi_k = \phi_k(t_k, \xi_k, v_k)$ be given by Proposition 3.3. Let $(x_k)_k$ be any sequence of points in $B_{\xi_k}(2\sqrt{\delta_k})$. There holds:*

- If $n \geq 7$:

$$\begin{aligned} \theta_k(x_k) |\nabla \phi_k(x_k)| + |\phi_k(x_k)| &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \delta_k \\ &+ \left[\delta_k^{\frac{n}{2}} + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \delta_k^2 \left| \ln \left(\frac{\theta_k(x_k)}{\delta_k} \right) \right| \right. \\ &\left. + \|h - c_n S_g\|_{L^\infty(2r_k)} \theta_k(x_k)^2 + \theta_k(x_k)^4 \mathbb{1}_{nlocf} \right] W_k(x_k) + \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2. \end{aligned} \quad (5.1)$$

- If $n = 6$:

$$\begin{aligned} \theta_k(x_k) |\nabla \phi_k(x_k)| + |\phi_k(x_k)| &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \delta_k \\ &+ \left[\delta_k^3 + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)} \left(\delta_k^2 \left| \ln \left(\frac{\theta_k(x_k)}{\delta_k} \right) \right| + \theta_k(x_k)^2 \right) \right] W_k(x_k). \end{aligned} \quad (5.2)$$

In (5.1) and (5.2) we have let:

$$\Omega_k = B_{\xi_k}(2r_k) \setminus B_{\xi_k}(\sqrt{\delta_k}), \quad (5.3)$$

and $\theta_k(x_k)$ is as in (4.25).

Proof. Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k \gg \mu_k^{\frac{3}{2}}$ as $k \rightarrow +\infty$, where μ_k is defined in (2.5). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, let $v_k \in F_k = F(\varepsilon_k, t_k, \xi_k)$ and let $\phi_k = \phi_k(t_k, \xi_k, v_k)$ be given by Proposition 3.3. Proposition 4.1 applies so that u_k given by (4.2) solves (4.3). As before, we shall omit the dependence in t_k and ξ_k and let $W_k = W_{k, t_k, \xi_k}$ and so on. We first obtain pointwise estimates on ϕ_k that we later improve into gradient estimates.

We first assume that $n \geq 7$. From (4.3), using (2.12) and (9.1) below, ϕ_k is easily seen to satisfy:

$$\begin{aligned} (\triangle_g + h) \left(\phi_k - \sum_{i=0}^n \lambda_k^i Z_{i,k} \right) &= f \left(u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right) \\ &+ (f - f(\xi_k)) W_k^{2^*-1} - (h - c_n S_g) W_k \\ &- c_n S_{g_{\xi_k}} \Lambda_{g_{\xi_k}}^{2^*-2} W_k + O(\delta_k^{\frac{n-2}{2}} \mathbb{1}_{d_k \leq 2r_k}) + O(\delta_k^{\frac{n-2}{2}} r_k^{-n} \mathbb{1}_{r_k \leq d_k \leq 2r_k}) \\ &+ (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(u_k^{-2^*-1} - u^{-2^*-1} \right) \\ &+ \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}}. \end{aligned} \quad (5.4)$$

Remember that by (2.15) there holds that $r_k \gg \sqrt{\delta_k}$ for $t \in [1/D, D]$, where δ_k is given by (2.16). In (5.4) we have let, for any $x \in M$:

$$d_k(x) = d_{g_{\xi_k}}(\xi_k, x). \quad (5.5)$$

Also, in (5.4), the notation $\delta_k^{\frac{n-2}{2}} r_k^{-n} \mathbb{1}_{r_k \leq d_k \leq 2r_k}$ is used to denote a smooth function, supported in $B_{\xi_k}(2r_k) \setminus B_{\xi_k}(r_k)$, and whose C^0 norm is bounded by $\delta_k^{\frac{n-2}{2}} r_k^{-n}$. Let $(x_k)_k$ be a sequence of points in $B_{\xi_k}(2\sqrt{\delta_k})$. If $x_k \in \Omega_k$ as in (5.3) we clearly have:

$$|\phi_k(x_k)| \leq \|\phi_k\|_{L^\infty(\Omega_k)}. \quad (5.6)$$

Assume now that $x_k \in B_{\xi_k}(\sqrt{\delta_k})$. As before, we let G denote the Green's function of $\Delta_g + h$ in M . A representation formula for $\phi_k - \sum_{i=0}^n \lambda_k^i Z_{i,k}$ in $B_{\xi_k}(2\sqrt{\delta_k})$ gives, using (2.15) and (5.4), that:

$$\begin{aligned} \phi_k(x_k) &= \sum_{i=0}^n \lambda_k^i Z_{i,k}(x_k) + O(\delta_k^{\frac{n-2}{2}} r_k^{2-n}) + O(\|\phi_k\|_{L^\infty(\Omega_k)}) \\ &\quad + O(\sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)}) + O\left(\sum_{i=0}^n |\lambda_k^i|\right) \\ &\quad + I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned} \quad (5.7)$$

where we have let:

$$\begin{aligned} I_1 &= \int_{B_{\xi_k}(2\sqrt{\delta_k})} f(y) \left(u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right) (y) G(x_k, y) dv_g(y), \\ I_2 &= \int_{B_{\xi_k}(2\sqrt{\delta_k})} (f(y) - f(\xi_k)) W_k^{2^*-1}(y) G(x_k, y) dv_g(y), \\ I_3 &= - \int_{B_{\xi_k}(2\sqrt{\delta_k})} (h - c_n S_g)(y) W_k(y) G(x_k, y) dv_g(y), \\ I_4 &= - \int_{B_{\xi_k}(2\sqrt{\delta_k})} c_n S_{g_{\xi_k}}(y) \Lambda_{g_{\xi_k}}^{2^*-2}(y) W_k(y) G(x_k, y) dv_g(y), \\ I_5 &= \int_{B_{\xi_k}(2\sqrt{\delta_k})} (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(u_k^{-2^*-1} - u^{-2^*-1} \right) (y) G(x_k, y) dv_g(y), \text{ and} \\ I_6 &= \int_{B_{\xi_k}(2\sqrt{\delta_k})} \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}} (y) G(x_k, y) dv_g(y). \end{aligned} \quad (5.8)$$

The definition of W_k in (2.17) yields:

$$I_2 \lesssim \delta_k \|\nabla f\|_{L^\infty(2r_k)} W_k(x_k), \quad (5.9)$$

while an application of Giraud's lemma (see [25], lemma 7.5) shows that

$$I_3 \lesssim \|h - c_n S_g\|_{L^\infty(2r_k)} \theta_k(x_k)^2 W_k(x_k) \quad (5.10)$$

and, using (2.4), that:

$$I_4 = \begin{cases} 0 & \text{if } (M, g) \text{ is locally conformally flat} \\ O(\theta_k(x_k)^4 W_k(x_k)) & \text{if } n \geq 7 \text{ and } (M, g) \text{ is not l.c.f.} \\ O(\delta_k^2 |\ln(\theta_k(x_k))|) & \text{if } n = 6 \text{ and } (M, g) \text{ is not l.c.f.} \end{cases} \quad (5.11)$$

where $\theta_k(x)$ is as in (4.25). Since u_k is uniformly bounded from below by (4.17) we also have:

$$I_5 \lesssim \delta_k. \quad (5.12)$$

With (9.11) below and since $v_k \in F_k$ (defined in (2.27)), we obtain that:

$$I_6 \lesssim \delta_k. \quad (5.13)$$

Note that (5.9), (5.11), (5.12) and (5.13) actually hold even if $n = 6$. By (4.1) there holds, in $B_{\xi_k}(\sqrt{\delta_k})$:

$$\left| u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right| \lesssim W_k^{2^*-2} |\phi_k| + W_k^{2^*-2}, \quad (5.14)$$

so that we have

$$I_1 \lesssim \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2 \left(\|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} + 1 \right). \quad (5.15)$$

Gathering the estimates (5.9) to (5.15) in (5.7) and using (2.15) gives:

$$\begin{aligned} \left| \phi_k(x_k) - \sum_{i=0}^n \lambda_k^i Z_{i,k}(x_k) \right| &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \sum_{i=0}^n |\lambda_k^i| \\ &\quad + \delta_k + \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2 \left(1 + \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \right) \\ &\quad + \left[\delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \theta_k(x_k)^2 + \theta_k(x_k)^4 \mathbf{1}_{nclf} \right] W_k(x_k). \end{aligned} \quad (5.16)$$

In (5.16), the notation $\mathbf{1}_{nclf}$ is used to denote a term which vanishes when (M, g) is conformally flat. Equation (5.16) holds for any sequence $x_k \in B_{\xi_k}(\sqrt{\delta_k})$. Applying it to a well-chosen set of $(n+1)$ points lying in $B_{\xi_k}(\delta_k)$ one obtains that:

$$\begin{aligned} \sum_{i=0}^n |\lambda_k^i| &\lesssim \delta_k^{\frac{n-2}{2}} \left(\|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} \right) \\ &\quad + \delta_k^{\frac{n-2}{2}} + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \delta_k^2 \\ &\quad + \delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} + \delta_k^4 \mathbf{1}_{nclf}. \end{aligned} \quad (5.17)$$

Plugging (5.17) into (5.16), using that $|Z_{i,k}| \lesssim W_k$ and using (5.6) gives the improved estimate:

$$\begin{aligned} |\phi_k(x_k)| &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} \\ &\quad + \delta_k + \left(1 + \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \right) \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2 \\ &\quad + \left[\delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \theta_k(x_k)^2 + \theta_k(x_k)^4 \mathbf{1}_{nclf} \right] W_k(x_k), \end{aligned} \quad (5.18)$$

which now holds for any sequence $x_k \in B_{\xi_k}(2\sqrt{\delta_k})$. We now claim that the following holds:

Claim 5.2. *There holds*

$$\|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \lesssim \max(1, M_k), \quad (5.19)$$

where we have let

$$\begin{aligned} M_k &= \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \delta_k \\ &\quad + \delta_k^{2-\frac{n}{2}} \|\nabla f\|_{L^\infty(2r_k)} + \delta_k^{3-\frac{n}{2}} \|h - c_n S_g\|_{L^\infty(2r_k)} + \delta_k^{5-\frac{n}{2}} \mathbf{1}_{nclf}, \end{aligned} \quad (5.20)$$

and where Ω_k is as in (5.3).

Proof of Claim 5.2. Let $(x_k)_k$ be a sequence of points of $B_{\xi_k}(2\sqrt{\delta_k})$ satisfying $|\phi_k(x_k)| = \max_{B_{\xi_k}(2\sqrt{\delta_k})} |\phi_k|$. Claim 5.2 is trivially satisfied if $x_k \in \Omega_k$, so we assume in the following that $x_k \in B_{\xi_k}(\sqrt{\delta_k})$. We proceed by contradiction and assume that there holds:

$$|\phi_k(x_k)| = \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \gg \max(1, M_k) \quad (5.21)$$

as $k \rightarrow +\infty$, where M_k is defined in (5.20). Let $(y_k)_k$ be any other sequence of points in $B_{\xi_k}(\sqrt{\delta_k})$. Assumption (5.21) implies in particular that $\|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \gg 1$, so that applying (5.18) at y_k yields, with (5.21):

$$|\phi_k(y_k)| \lesssim \left(\left(\frac{\delta_k}{\theta_k(y_k)} \right)^2 + o(1) \right) \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}. \quad (5.22)$$

Using (5.22), we now compute again I_1 in (5.8) and obtain, since $\delta_k \leq \theta_k(y_k)$:

$$I_1 \lesssim \left(\left(\frac{\delta_k}{\theta_k(y_k)} \right)^{\frac{7}{2}} + o(1) \right) \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} + \left(\frac{\delta_k}{\theta_k(y_k)} \right)^2,$$

which in turn, with (5.7) and (5.21) implies that there holds:

$$|\phi_k(y_k)| \lesssim \left(\left(\frac{\delta_k}{\theta_k(y_k)} \right)^{\frac{7}{2}} + o(1) \right) \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}.$$

Replugging the latter estimate in the computation of I_1 improves again, and after a finite number of iterations one obtains:

$$|\phi_k(y_k)| \lesssim \left(\left(\frac{\delta_k}{\theta_k(y_k)} \right)^{n-2} + o(1) \right) \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \text{ for any } y_k \in B_{\xi_k}(\sqrt{\delta_k}). \quad (5.23)$$

In particular, (5.23) applied to the sequence x_k given by (5.21) yields:

$$\theta_k(x_k) \lesssim \delta_k,$$

where θ_k is as in (4.25). We define now, for $y \in B_0(2\delta_k^{-\frac{1}{2}})$:

$$\tilde{\phi}_k(x) = \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}^{-1} \phi_k \left(\exp^{g_{\xi_k}}(\delta_k y) \right),$$

and let $\tilde{x}_k = \frac{1}{\delta_k} \left(\exp^{g_{\xi_k}} \right)^{-1}(x_k)$. Hence, $\tilde{x}_k \rightarrow \tilde{x}_0 \in \mathbb{R}^n$ as $k \rightarrow +\infty$. We also have $\|\tilde{\phi}_k\|_{L^\infty(2\delta_k^{-\frac{1}{2}})} \leq 1$, and using (5.4), (4.1), (5.21) and standard elliptic theory, we obtain that $\tilde{\phi}_k$ converges in $C_{loc}^1(\mathbb{R}^n)$, up to a subsequence, to some function $\tilde{\phi}_0$ satisfying $|\tilde{\phi}_0(\tilde{x}_0)| = 1$ and

$$\Delta_\xi \tilde{\phi}_0 - (2^* - 1)f(\xi_0)U_{\xi_0}^{2^*-2}\tilde{\phi}_0 = \sum_{i=0}^n \tilde{\lambda}_i \Delta_\xi V_{i,\xi_0}, \quad (5.24)$$

where $\xi_0 = \lim_{k \rightarrow +\infty} \xi_k$, U_{ξ_0} is defined in (2.20), V_{i,ξ_0} is as in (2.18) and where we have let, up to a subsequence:

$$\tilde{\lambda}_i = \lim_{k \rightarrow +\infty} \frac{\lambda_k^i}{\delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}}.$$

Note that this limit exists, up to a subsequence, by (5.17) and (5.21). Also, in (5.24), ξ stands for the Euclidean metric in \mathbb{R}^n . Passing (5.23) to the limit also gives that:

$$|\tilde{\phi}_0(y)| \lesssim (1 + |y|)^{2-n} \text{ for any } y \in \mathbb{R}^n. \quad (5.25)$$

With the latter estimate we can integrate (5.24) against V_{i,ξ_0} for any $0 \leq i \leq n$. Since V_{i,ξ_0} solves (2.19) there holds $\tilde{\lambda}_i = 0$ and thus $\tilde{\phi}_0$ satisfies:

$$\Delta_\xi \tilde{\phi}_0 = (2^* - 1)f(\xi_0)U_{\xi_0}^{2^*-2}\tilde{\phi}_0, \quad (5.26)$$

Again with (5.25) this shows that $\tilde{\phi}_0 \in H^1(M)$ and then the Bianchi-Egnell [5] classification result applies and gives that

$$\tilde{\phi}_0 \in \text{Span}\{V_{i,\xi_0}, 0 \leq i \leq n\}. \quad (5.27)$$

To conclude the proof of Claim 5.2 we now show that $\tilde{\phi}_0 \in \text{Span}\{V_{i,\xi}, 0 \leq i \leq n\}^\perp$. With (5.27) this will show that $\tilde{\phi}_0 \equiv 0$, thus contradicting the fact that $|\tilde{\phi}_0(\tilde{x}_0)| = 1$. By (3.8), $\phi_k \in K_k^\perp$, where $K_k = K_{k,t_k,\xi_k}$ is defined in (2.22). Hence for any $0 \leq i \leq n$ there holds:

$$\int_M \left(\langle \nabla Z_{i,k}, \nabla \phi_k \rangle_g + h Z_{i,k} \phi_k \right) dv_g = 0.$$

Let $R > 0$ and $0 \leq i \leq n$. Integrating by parts the latter equation gives

$$\begin{aligned} \int_{B_{\xi_k}(R\delta_k)} \langle \nabla Z_{i,k}, \nabla \phi_k \rangle_g + h Z_{i,k} \phi_k dv_g &= - \int_{\partial B_{\xi_k}(R\delta_k)} \phi_k \partial_\nu Z_{i,k} d\sigma_g \\ &- \int_{M \setminus B_{\xi_k}(R\delta_k)} (h - c_n S_g) Z_{i,k} \phi_k dv_g - \int_{M \setminus B_{\xi_k}(R\delta_k)} (\Delta_g + c_n S_g) Z_{i,k} \phi_k dv_g. \end{aligned} \quad (5.28)$$

Using (3.9) and Hölder's inequality we get, with (5.21), that:

$$\begin{aligned} \left| \int_{M \setminus B_{\xi_k}(R\delta_k)} (h - c_n S_g) Z_{i,k} \phi_k dv_g \right| &\lesssim \|h - c_n S_g\|_{L^\infty(2r_k)} \eta \delta_k^2 \varepsilon_k \\ &= o\left(\delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right). \end{aligned} \quad (5.29)$$

Using (5.23) we have:

$$\left| \int_{\partial B_{\xi_k}(R\delta_k)} \phi_k \partial_\nu Z_{i,k} d\sigma_g \right| \lesssim \left(\frac{1}{(1+R)^{n-2}} + o(1) \right) \delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}. \quad (5.30)$$

We now compute the third integral in (5.28) by using (9.2) below. By Proposition 4.1 and (2.15) we have

$$\begin{aligned} \int_{M \setminus B_{\xi_k}(R\delta_k)} \delta_k^{\frac{n}{2}} r_k^{-n-1} \mathbb{1}_{r_k \leq d_k \leq 2r_k} |\phi_k| dv_g &+ \int_{M \setminus B_{\xi_k}(R\delta_k)} \delta_k^{\frac{n-2}{2}} r_k^{-n} \mathbb{1}_{r_k \leq d_k \leq 2r_k} |\phi_k| dv_g \\ &+ \int_{M \setminus B_{\xi_k}(R\delta_k)} \delta_k^{\frac{n-2}{2}} \phi_k dv_g = o(\delta_k^{\frac{n-2}{2}}) = o\left(\delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right), \end{aligned} \quad (5.31)$$

where the last equality is due to (5.21). By (2.4) there holds

$$|S_{g_{\xi_k}}|(x) \lesssim d_k(x)^2 \text{ in } B_{\xi_k}(2r_k),$$

where d_k is the Riemannian distance defined in (5.5), so that by (5.21) and (5.23) we have:

$$\begin{aligned} \int_{M \setminus B_{\xi_k}(R\delta_k)} c_n S_{g_{\xi_k}} \Lambda_{\xi_k}^{2^*-2} Z_{i,k} \phi_k dv_g &= \int_{B_{\xi_k}(2r_k) \setminus B_{\xi_k}(\sqrt{\delta_k})} c_n S_{g_{\xi_k}} \Lambda_{\xi_k}^{2^*-2} Z_{i,k} \phi_k dv_g \\ &+ \int_{B_{\xi_k}(\sqrt{\delta_k}) \setminus B_{\xi_k}(R\delta_k)} c_n S_{g_{\xi_k}} \Lambda_{\xi_k}^{2^*-2} Z_{i,k} \phi_k dv_g \\ &= o\left(\delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right), \end{aligned} \quad (5.32)$$

where the last equality is again given by (5.21). In case where (M, g) is not locally conformally flat we have, using (5.21), that:

$$\begin{aligned} \left| \int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} \delta_k^{\frac{n}{2}} \theta_k(\cdot)^{2-n} \mathbf{1}_{d_k \leq 2r_k} |\phi_k| dv_g \right| &\lesssim \delta_k^{\frac{n}{2}} \|\phi_k\|_{L^\infty(\Omega_k)} \\ &= o\left(\delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right), \end{aligned} \quad (5.33)$$

and using (5.23) we also obtain that:

$$\left| \int_{B_{\xi_k}(\sqrt{\delta_k}) \setminus B_{\xi_k}(R\delta_k)} \delta_k^{\frac{n}{2}} \theta_k(\cdot)^{2-n} \mathbf{1}_{d_k \leq 2r_k} |\phi_k| dv_g \right| = o\left(\delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right). \quad (5.34)$$

Finally, by (5.21) we can write that:

$$\begin{aligned} \left| \int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} f(\xi_k) W_k^{2^*-2} Z_{i,k} \phi_k dv_g \right| &\lesssim \delta_k^{\frac{n}{2}} \|\phi_k\|_{L^\infty(\Omega_k)}, \\ &= o\left(\delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right) \end{aligned} \quad (5.35)$$

where Ω_k is as in (5.3), and using (5.23) one gets that

$$\begin{aligned} \left| \int_{B_{\xi_k}(\sqrt{\delta_k}) \setminus B_{\xi_k}(R\delta_k)} f(\xi_k) W_k^{2^*-2} Z_{i,k} \phi_k dv_g \right| \\ \lesssim \left(\frac{1}{(1+R)^{n-2}} + o(1) \right) \delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}. \end{aligned} \quad (5.36)$$

Combining (5.31), (5.32), (5.33), (5.34), (5.35) and (5.36) in (9.2) we obtain that:

$$\left| \int_{M \setminus B_{\xi_k}(R\delta_k)} (\Delta_g + c_n S_g) Z_{i,k} \phi_k dv_g \right| \lesssim \left(\frac{1}{(1+R)^{n-2}} + o(1) \right) \delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}. \quad (5.37)$$

We now divide (5.28) by $\delta_k^{\frac{n-2}{2}} \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}$. We first use (5.29), (5.30), (5.37), the C^1 convergence of $\tilde{\phi}_k$ towards $\tilde{\phi}_0$ and the expression of $Z_{i,k}$ in (2.21) to pass to the limit as $k \rightarrow +\infty$. We then use (5.25) to pass to the limit as $R \rightarrow +\infty$, to obtain that:

$$\int_{\mathbb{R}^n} \langle \nabla \tilde{\phi}_0, \nabla V_{i,\xi_0} \rangle_{eucl} dx = 0 \text{ for all } 0 \leq i \leq n,$$

where $\xi_0 = \lim_{k \rightarrow +\infty} \xi_k$. We thus have in the end that:

$$\tilde{\phi}_0 \in \text{Span}\{V_{i,\xi_0}, 0 \leq i \leq n\}^\perp.$$

With (5.27), this implies that $\tilde{\phi}_0 \equiv 0$ which contradicts the fact that $|\tilde{\phi}_0(\tilde{x}_0)| = 1$, and therefore concludes the proof of Claim 5.2. \square

Hence, equation (5.19) holds true. Assume first that there holds $M_k \lesssim 1$, where M_k is defined in (5.20). Then (5.18) shows that, for any sequence $x_k \in B_{\xi_k}(2\sqrt{\delta_k})$, there holds:

$$\begin{aligned} |\phi_k(x_k)| &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \delta_k + \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2 \\ &\quad + \left[\delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \theta_k(x_k)^2 + \theta_k(x_k)^4 \mathbf{1}_{n \leq 4} \right] W_k(x_k). \end{aligned} \quad (5.38)$$

Assume now that $M_k \gg 1$. Then, plugging (5.20) in (5.18) yields:

$$\begin{aligned}
|\phi_k(x_k)| &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \delta_k + \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2 \\
&+ \left[\delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \theta_k(x_k)^2 + \theta_k(x_k)^4 \mathbf{1}_{n l c f} \right] W_k(x_k) \\
&+ \left[\delta_k^3 + \delta_k^{4-\frac{n}{2}} \|\nabla f\|_{L^\infty(2r_k)} + \delta_k^{5-\frac{n}{2}} \|h - c_n S_g\|_{L^\infty(2r_k)} \right. \\
&\quad \left. + \delta_k^{7-\frac{n}{2}} \mathbf{1}_{n l c f} \right] \theta_k(x_k)^{-2}.
\end{aligned} \tag{5.39}$$

We proceed again as we did to obtain (5.23): we use (5.39) to obtain a better estimate of I_1 in (5.8), which in turn gives with (5.7) an improved pointwise estimate of $|\phi_k|$. After a finite number of iterations one obtains:

$$\begin{aligned}
|\phi_k(x_k)| &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \delta_k \\
&+ \left[\delta_k^{\frac{n}{2}} + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \delta_k^2 \left| \ln \left(\frac{\theta_k(x_k)}{\delta_k} \right) \right| \right. \\
&\quad \left. + \|h - c_n S_g\|_{L^\infty(2r_k)} \theta_k(x_k)^2 + \theta_k(x_k)^4 \mathbf{1}_{n l c f} \right] W_k(x_k) + \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2.
\end{aligned} \tag{5.40}$$

We now assume that $n = 6$. The proof closely follows the $n \geq 7$ case so we only highlight the main differences. From (4.3), using (2.12), Proposition 4.1 and (9.1), it is easily seen that ϕ_k satisfies:

$$\begin{aligned}
(\triangle_g + h) \left(\phi_k - \sum_{i=0}^6 \lambda_k^i Z_{i,k} \right) &= (2 + o(1)) f(u + W_k) \phi_k \\
&+ (f - f(\xi_k)) W_k^2 - \left(h - \frac{1}{5} S_g - 2fu \right) W_k \\
&- \frac{1}{5} S_{g_{\xi_k}} \Lambda_{g_{\xi_k}} W_k + O(\delta_k^2 \mathbf{1}_{d_k \leq 2r_k}) + O(\delta_k^2 r_k^{-6} \mathbf{1}_{r_k \leq d_k \leq 2r_k}) \\
&+ (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) (u_k^{-4} - u^{-4}) \\
&+ \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^4}.
\end{aligned} \tag{5.41}$$

Let $(x_k)_k$ be a sequence of points in $B_{\xi_k}(2\sqrt{\delta_k})$. If $x_k \in \Omega_k$ we have as before:

$$|\phi_k(x_k)| \leq \|\phi_k\|_{L^\infty(\Omega_k)}. \tag{5.42}$$

Assume now that $x_k \in B_{\xi_k}(\sqrt{\delta_k})$. Mimicking (5.10), there holds

$$\int_{B_{\xi_k}(2\sqrt{\delta_k})} \left(h - \frac{1}{5} S_g - 2fu \right) W_k(y) G(x_k, y) dv_g(y) \lesssim \|h - \frac{1}{5} S_g - 2fu\|_{L^\infty(2r_k)} \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2,$$

so that using (5.9)–(5.13), a representation formula for $\phi_k - \sum_{i=0}^6 \lambda_k^i Z_{i,k}$ in $B_{\xi_k}(2\sqrt{\delta_k})$ gives, with (5.41), that:

$$\begin{aligned} \left| \phi_k - \sum_{i=0}^6 \lambda_k^i Z_{i,k} \right| (x_k) &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \sum_{i=0}^6 |\lambda_k^i| + \delta_k \\ &+ \delta_k \|\nabla f\|_{L^\infty(2r_k)} W_k(x_k) + \left(\left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)} + \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \right) \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2. \end{aligned} \quad (5.43)$$

As before, the $|\lambda_k^i|$ are estimated by

$$\begin{aligned} \sum_{i=0}^6 |\lambda_k^i| &\lesssim \delta_k^2 \left(\|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} \right) + \delta_k^3 \\ &+ \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)} \delta_k^2 + \delta_k^2 \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}, \end{aligned} \quad (5.44)$$

so that plugging (5.44) in (5.43) and using (5.42) gives, for any sequence of points $x_k \in B_{\xi_k}(2\sqrt{\delta_k})$:

$$\begin{aligned} |\phi_k(x_k)| &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \delta_k \|\nabla f\|_{L^\infty(2r_k)} W_k(x_k) + \delta_k \\ &+ \left(\left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)} + \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \right) \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2. \end{aligned} \quad (5.45)$$

We now claim that there holds:

$$\|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \lesssim N_k \quad (5.46)$$

where we have let:

$$N_k = \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \delta_k + \delta_k^{-1} \|\nabla f\|_{L^\infty(2r_k)} + \left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)}, \quad (5.47)$$

and where Ω_k is as in (5.3). To prove (5.46), let $(x_k)_k$ be a sequence of points in $B_{\xi_k}(2\sqrt{\delta_k})$ satisfying $|\phi_k(x_k)| = \max_{B_{\xi_k}(2\sqrt{\delta_k})} |\phi_k|$. Estimate (5.46) is trivially satisfied if $x_k \in \Omega_k$, so we assume in the following that $x_k \in B_{\xi_k}(\sqrt{\delta_k})$. We proceed by contradiction and assume that there holds:

$$|\phi_k(x_k)| = \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \gg N_k \quad (5.48)$$

as $k \rightarrow +\infty$, where N_k is defined in (5.47). Let $(y_k)_k$ be any other sequence of points in $B_{\xi_k}(\sqrt{\delta_k})$. Proceeding as in (5.22)–(5.23) we obtain that there holds:

$$|\phi_k(y_k)| \lesssim \left(\left(\frac{\delta_k}{\theta_k(y_k)} \right)^4 + o(1) \right) \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}, \quad (5.49)$$

so that (5.49) applied to the sequence x_k given by (5.48) yields:

$$\theta_k(x_k) \lesssim \delta_k,$$

where θ_k is as in (4.25). Define now, for $y \in B_0(2\delta_k^{-\frac{1}{2}})$:

$$\tilde{\phi}_k(x) = \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}^{-1} \phi_k \left(\exp^{g_{\xi_k}}(\delta_k y) \right),$$

and let $\tilde{x}_k = \frac{1}{\delta_k} \left(\exp^{g_{\xi_k}} \right)^{-1} (x_k)$. Here again, $\tilde{x}_k \rightarrow \tilde{x}_0 \in \mathbb{R}^n$ as $k \rightarrow +\infty$. The arguments that led to (5.27) adapt therefore with no modification and we obtain in the end that $\tilde{\phi}_k$ converges in $C_{loc}^1(\mathbb{R}^n)$, up to a subsequence, to some function $\tilde{\phi}_0$ satisfying $|\tilde{\phi}_0(\tilde{x}_0)| = 1$ and

$$\tilde{\phi}_0 \in \text{Span}\{V_{i,\xi_0}, 0 \leq i \leq 6\}. \quad (5.50)$$

To conclude the proof of (5.46) we now show that $\tilde{\phi}_0 \in \text{Span}\{V_{i,\xi}, 0 \leq i \leq 6\}^\perp$. By (3.8), $\phi_k \in K_k^\perp$, where $K_k = K_{k,t_k,\xi_k}$ is defined in (2.22). Hence for any $0 \leq i \leq 6$ and for any $R > 0$ fixed, there holds:

$$\begin{aligned} \int_{B_{\xi_k}(R\delta_k)} \langle \nabla Z_{i,k}, \nabla \phi_k \rangle_g + h Z_{i,k} \phi_k dv_g &= - \int_{\partial B_{\xi_k}(R\delta_k)} \phi_k \partial_\nu Z_{i,k} d\sigma_g \\ &- \int_{M \setminus B_{\xi_k}(R\delta_k)} \left(h - \frac{1}{5} S_g \right) Z_{i,k} \phi_k dv_g - \int_{M \setminus B_{\xi_k}(R\delta_k)} \left(\Delta_g + \frac{1}{5} S_g \right) Z_{i,k} \phi_k dv_g. \end{aligned} \quad (5.51)$$

By (5.48) and (2.21) one has that:

$$\int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} \left(h - \frac{1}{5} S_g \right) Z_{i,k} \phi_k dv_g = o(\delta_k^2 \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}),$$

while using (5.49) there holds that:

$$\int_{B_{\xi_k}(\sqrt{\delta_k}) \setminus B_{\xi_k}(R\delta_k)} (h - c_n S_g) Z_{i,k} \phi_k dv_g = o(\delta_k^2 \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}),$$

that

$$\left| \int_{\partial B_{\xi_k}(R\delta_k)} \phi_k \partial_\nu Z_{i,k} d\sigma_g \right| \lesssim \left(\frac{1}{(1+R)^4} + o(1) \right) \delta_k^2 \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))},$$

and, with (2.15) and (5.48), that

$$\begin{aligned} \int_{M \setminus B_{\xi_k}(R\delta_k)} \delta_k^3 r_k^{-7} \mathbf{1}_{r_k \leq d_k \leq 2r_k} |\phi_k| dv_g + \int_{M \setminus B_{\xi_k}(R\delta_k)} \delta_k^2 r_k^{-6} \mathbf{1}_{r_k \leq d_k \leq 2r_k} |\phi_k| dv_g \\ + \int_{M \setminus B_{\xi_k}(R\delta_k)} \delta_k^2 \phi_k dv_g = o\left(\delta_k^2 \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right). \end{aligned}$$

Using (2.4), (5.48) and (5.49) we get also that:

$$\begin{aligned} \int_{M \setminus B_{\xi_k}(R\delta_k)} \frac{1}{5} S_{g_{\xi_k}} \Lambda_{g_{\xi_k}} Z_{i,k} \phi_k dv_g \\ = \int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} \frac{1}{5} S_{g_{\xi_k}} \Lambda_{g_{\xi_k}} Z_{i,k} \phi_k dv_g + \int_{B_{\xi_k}(\sqrt{\delta_k}) \setminus B_{\xi_k}(R\delta_k)} \frac{1}{5} S_{g_{\xi_k}} \Lambda_{g_{\xi_k}} Z_{i,k} \phi_k dv_g \\ = o\left(\delta_k^2 \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right). \end{aligned}$$

Similarly, we get that

$$\int_{M \setminus B_{\xi_k}(R\delta_k)} \delta_k^3 \theta_k(\cdot)^{-4} \mathbf{1}_{d_k \leq 2r_k} |\phi_k| dv_g = o\left(\delta_k^2 \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right)$$

that

$$\begin{aligned} \left| \int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} f(\xi_k) W_k Z_{i,k} \phi_k dv_g \right| &\lesssim \delta_k^3 \|\phi_k\|_{L^\infty(\Omega_k)}, \\ &= o\left(\delta_k^2 \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))}\right), \end{aligned}$$

and, using (5.49), that

$$\left| \int_{B_{\xi_k}(\sqrt{\delta_k}) \setminus B_{\xi_k}(R\delta_k)} f(\xi_k) W_k Z_{i,k} \phi_k dv_g \right| \lesssim \left(\frac{1}{(1+R)^4} + o(1) \right) \delta_k^2 \|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))},$$

so that passing to the limit as $k \rightarrow +\infty$ and then as $R \rightarrow +\infty$ gives, with (5.51):

$$\tilde{\phi}_0 \in \text{Span}\{V_{i,\xi_0}, 0 \leq i \leq 6\}^\perp,$$

which is a contradiction with (5.50) since $|\tilde{\phi}_0(\tilde{x}_0)| = 1$. Therefore (5.46) is proven, and an iteration argument as for the $n \geq 7$ case gives:

$$\begin{aligned} |\phi_k(x_k)| &\lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \delta_k + \left[\delta_k^3 + \delta_k \|\nabla f\|_{L^\infty(2r_k)} \right. \\ &\quad \left. + \left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)} \delta_k^2 \left| \ln \left(\frac{\theta_k(x_k)}{\delta_k} \right) \right| + \left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)} \theta_k(x_k)^2 \right] W_k(x_k). \end{aligned} \quad (5.52)$$

To conclude the proof of Proposition 5.1 it remains to improve (5.40) and (5.52) into the final estimates (5.1) and (5.2). We only prove the $n \geq 7$ case, as the six-dimensional case follows from similar arguments. Let $(x_k)_k$ be any sequence of points in $B_{\xi_k}(2\sqrt{\delta_k})$. We write a Green representation formula and differentiate it at x_k . As in (5.9)–(5.13) one obtains that:

$$\begin{aligned} \left| \nabla \left(\psi_k - \sum_{j=0}^n \lambda_k^j Z_{i,k} \right) (x_k) \right| &\lesssim \frac{1}{\sqrt{\delta_k}} \|\phi_k\|_{L^\infty(\Omega_k)} + \|\nabla \phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \\ &\quad + \left[\delta_k^{\frac{n-2}{2}} + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \theta_k(x_k)^2 + \theta_k(x_k)^4 \mathbb{1}_{n \leq 6} \right] \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{1-n} \\ &\quad + \int_{B_{\xi_k}(2\sqrt{\delta_k})} f \left| u_k^{2^*-1} - W_k^{2^*-1} - u^{2^*-1} \right| (y) \nabla_x G(x_k, y) dv_g(y). \end{aligned} \quad (5.53)$$

The last term is then estimated with (5.40) to obtain (5.1). \square

Note that using Claim 5.2 and (5.46) in (5.17) and (5.44) yields:

- If $n \geq 7$:

$$\begin{aligned} \sum_{i=0}^n |\lambda_k^i| &\lesssim \delta_k^{\frac{n-2}{2}} \left(\|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} \right) + \delta_k^{\frac{n-2}{2}} \\ &\quad + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \delta_k^2 + \delta_k^4 \mathbb{1}_{n \leq 6}. \end{aligned} \quad (5.54)$$

- If $n = 6$:

$$\begin{aligned} \sum_{i=0}^6 |\lambda_k^i| &\lesssim \delta_k^2 \left(\|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)} \right) + \delta_k^3 \\ &\quad + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)} \delta_k^2. \end{aligned} \quad (5.55)$$

As a consequence of these refined local estimates we are now in position to obtain global pointwise estimates on ϕ_k in the whole manifold M :

Proposition 5.3. *Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k \gg \mu_k^{\frac{3}{2}}$ as $k \rightarrow +\infty$, where μ_k is defined in (2.5). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, let $v_k \in F_k = F(\varepsilon_k, t_k, \xi_k)$ and let $\phi_k = \phi_k(t_k, \xi_k, v_k)$ be given by Proposition 3.3. Let $(x_k)_k$ be any sequence of points in M . There holds then:*

$$|\phi_k(x_k)| \leq C (\delta_k + \eta \varepsilon_k) \left(u(x_k) + W_k(x_k) \right), \quad (5.56)$$

where η is as in (2.11), for some positive constant C independent of η and k . As a consequence, we have the following gradient estimate:

$$|\nabla \phi_k(x_k)| \leq C (\delta_k + \eta \varepsilon_k) \left(1 + \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{1-n} \right).$$

Note that the constant C appearing in the statement of Proposition 5.3 *a priori* depends on g, u_0 (as in (2.8)), h, f, σ, π , but does not depend on the choice of the sequences $(\varepsilon_k)_k, (t_k)_k, (\xi_k)_k$ and $(v_k)_k$.

Proof. Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k \gg \mu_k^{\frac{3}{2}}$ as $k \rightarrow +\infty$, where μ_k is defined in (2.5). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, let $v_k \in F_k = F(\varepsilon_k, t_k, \xi_k)$ and let $\phi_k = \phi_k(t_k, \xi_k, v_k)$ be given by Proposition 3.3. Define:

$$\nu_k = \left\| \frac{\phi_k}{u + W_k} \right\|_{L^\infty(M)}. \quad (5.57)$$

Proposition 4.1 shows that $\nu_k = o(1)$ as $k \rightarrow +\infty$. We let in the following, for any $v \in H^1(M)$:

$$L_u(v) = \Delta_g v + \left[h - (2^* - 1)fu^{2^*-2} + (2^* + 1) \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+2}} \right] v. \quad (5.58)$$

L_u is the linearized operator of the scalar equation of (2.12) at u .

As before, assume first that $n \geq 7$. It is easily seen from (5.4) that ϕ_k satisfies:

$$\begin{aligned} L_u \phi_k &= \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k} + f \left(u_k^{2^*-1} - W_k^{2^*-1} - (2^* - 1)u^{2^*-2} \phi_k - u^{2^*-1} \right) \\ &\quad + (f - f(\xi_k)) W_k^{2^*-1} - (h - c_n S_g) W_k \\ &\quad - c_n S_{g_{\xi_k}} \Lambda_{g_{\xi_k}}^{2^*-2} W_k + O(\delta_k^{\frac{n-2}{2}} \mathbf{1}_{d_k \leq 2r_k}) + O(\delta_k^{\frac{n-2}{2}} r_k^{-n} \mathbf{1}_{r_k \leq d_k \leq 2r_k}) \\ &\quad + (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(u_k^{-2^*-1} - u^{-2^*-1} + (2^* + 1)u^{-2^*-2} \phi_k \right) \\ &\quad + \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}}. \end{aligned} \quad (5.59)$$

Let $(x_k)_k$ be any sequence of points in M . Let G_u be the Green's function of the operator L_u defined in (5.58). By (2.13), the operator L_u is coercive, and therefore its Green function satisfies (see for instance Robert [43]):

$$\frac{1}{C} d_{g_{\xi_k}}(x, y)^{2-n} \leq G_u(x, y) \leq C d_{g_{\xi_k}}(x, y)^{2-n}, \quad (5.60)$$

for some positive constant C . Note that since we assumed $|\sigma|_g + \pi > 0$ somewhere in M , the constant C in (5.60) does not depend on η in (2.11), provided η is small enough. First, by (5.54), by Proposition 4.1 and since $|Z_{i,k}| \lesssim W_k$ for all $0 \leq i \leq n$, we have:

$$\begin{aligned} \int_M G_u(x_k, y) \sum_{i=0}^n \lambda_k^i (\Delta_g + h) Z_{i,k}(y) dv_g(y) \\ \lesssim \left[\delta_k^{\frac{n-2}{2}} + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \delta_k^2 + \delta_k^4 \mathbf{1}_{n l c f} \right] \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{2-n}. \end{aligned} \quad (5.61)$$

Independently, straightforward computations using (5.60) show that there holds:

$$\begin{aligned} \int_M G_u(x_k, y) \left[(f(y) - f(\xi_k)) W_k^{2^*-1}(y) + (h - c_n S_g) W_k(y) + c_n S_{g_{\xi_k}}(y) W_k(y) \right] dv_g(y) \\ \lesssim \left(\delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \theta_k(x_k)^2 \right) \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{2-n} + \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{6-n} \mathbf{1}_{n l c f}. \end{aligned} \quad (5.62)$$

Similarly, there holds:

$$\int_M G_u(x_k, y) \left[\delta_k^{\frac{n-2}{2}} + \delta_k^{\frac{n-2}{2}} r_k^{-n} \mathbf{1}_{r_k \leq d_k \leq 2r_k} \right] dv_g(y) = O(\delta_k^{\frac{n-2}{2}} r_k^{2-n}), \quad (5.63)$$

where d_k is as in (5.5). Now, with (4.2), straightforward computations using Proposition 4.1 yield that:

$$\begin{aligned} & \int_{B_{\xi_k}(\sqrt{\delta_k})} G_u(x_k, y) (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \\ & \times \left(u_k^{-2^*-1} - u^{-2^*-1} + (2^* + 1)u^{-2^*-2}\phi_k \right) (y) dv_g(y) \lesssim \delta_k^{\frac{n}{2}} \theta_k(x_k)^{2-n} + \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{4-n}, \end{aligned} \quad (5.64)$$

while using (4.2), Proposition 4.1 and the definition of ν_k in (5.57) there holds:

$$\begin{aligned} & \int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} G_u(x_k, y) (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \\ & \times \left(u_k^{-2^*-1} - u^{-2^*-1} + (2^* + 1)u^{-2^*-2}\phi_k \right) (y) dv_g(y) \\ & \lesssim \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{4-n} + \nu_k^2. \end{aligned} \quad (5.65)$$

Similarly, using (4.2) and the definition of ν_k in (5.57) one obtains that:

$$\begin{aligned} & \int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} G_u(x_k, y) f \left(u_k^{2^*-1} - W_k^{2^*-1} - (2^* - 1)u^{2^*-2}\phi_k - u^{2^*-1} \right) (y) dv_g(y) \\ & \lesssim \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{4-n} + \nu_k^2. \end{aligned} \quad (5.66)$$

Using Proposition 5.1 and estimating $\|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(\Omega_k)}$ with Proposition 4.1 gives that:

$$\begin{aligned} & \int_{B_{\xi_k}(\sqrt{\delta_k})} G_u(x_k, y) f \left(u_k^{2^*-1} - W_k^{2^*-1} - (2^* - 1)u^{2^*-2}\phi_k - u^{2^*-1} \right) (y) dv_g(y) \\ & \lesssim \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2 + \left[\delta_k^{\frac{n}{2}} + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)} \delta_k^2 |\ln(\delta_k)| \right. \\ & \left. + \delta_k^2 \theta_k(x_k) \mathbb{1}_{n \leq f} \right] \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{2-n}. \end{aligned} \quad (5.67)$$

Finally, using again (9.11) below and since $v_k \in F_k$, there holds that:

$$\int_{B_{\xi_k}(\sqrt{\delta_k})} G_u(x_k, y) \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}} (y) dv_g(y) \lesssim \delta_k^{\frac{n}{2}} \theta_k(x_k)^{2-n} + \delta_k \varepsilon_k \quad (5.68)$$

and that

$$\int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} G_u(x_k, y) \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}} (y) dv_g(y) \lesssim \delta_k^{\frac{n}{2}} \theta_k(x_k)^{2-n} + \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{4-n} + \eta \varepsilon_k, \quad (5.69)$$

where η is as in (2.11). Writing a Green's representation formula for (5.59) together with (5.61) – (5.69), with Proposition 4.1 and with (2.15) gives then:

$$|\phi_k(x_k)| \lesssim \eta \varepsilon_k + \nu_k^2 + \delta_k \left(u(x_k) + W_k(x_k) \right) \quad (5.70)$$

where ν_k is as in (5.57). Also, to obtain (5.70) we used that there holds, for any $x \in M$:

$$\left(\frac{\delta_k}{\theta_k(x_k)} \right)^2 + \delta_k^{\frac{n-2}{2}} \theta_k(x_k)^{4-n} + \delta_k^{\frac{n}{2}} \theta_k(x_k)^{2-n} \leq C \delta_k \left(u(x) + W_k(x) \right), \quad (5.71)$$

for some positive constant C independent of η and k . Coming back to the definition of ν_k in (5.57), it remains to apply (5.70) at the sequence $(x_k)_k$ of points of M where ν_k is attained. Since, by Proposition 4.1, there holds that $\nu_k \rightarrow 0$ as $k \rightarrow +\infty$, we obtain in the end that

$$\nu_k \lesssim \delta_k + \eta \varepsilon_k,$$

which concludes the proof of (5.56) when $n \geq 7$.

Assume now that $n = 6$. Rewrite (5.4) as:

$$\begin{aligned} L_u \phi_k &= \sum_{i=0}^6 \lambda_k^i (\triangle_g + h) Z_{i,k} + 2fW_k \phi_k + f\phi_k^2 + (f - f(\xi_k))W_k^2 \\ &\quad - \left(h - \frac{1}{5}S_g - 2fu\right)W_k - \frac{1}{5}S_{g_{\xi_k}} \Lambda_{\xi_k} W_k + O(\delta_k^2 \mathbf{1}_{dk \leq 2r_k}) + O(\delta_k^2 r_k^{-6} \mathbf{1}_{r_k \leq dk \leq 2r_k}) \\ &\quad + (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) (u_k^{-4} - u^{-4} + 4u^{-5} \phi_k) \\ &\quad + \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^4}. \end{aligned} \quad (5.72)$$

Let $(x_k)_k$ be any sequence of points in M . As before, a Green's representation formula for (5.72) using Propositions 4.1 and 5.1 and (2.15) gives now the following estimate:

$$\begin{aligned} |\phi_k(x_k)| &\lesssim \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2 + \delta_k + \varepsilon_k + \nu_k^2 + \left(\delta_k + \delta_k^2 |\ln \delta_k| \|h - \frac{1}{5}S_g - 2fu\|_{L^\infty(2r_k)} + \nu_k^2 \right) \delta_k^2 \theta_k(x_k)^{-4} \\ &\lesssim (\delta_k + \eta \varepsilon_k + \nu_k^2) (u + W_k)(x_k). \end{aligned} \quad (5.73)$$

Applying (5.73) at the sequence of points where ν_k as in (5.57) is attained concludes the proof of (5.56) for the 6-dimensional case.

The gradient estimates in Proposition 5.3 are obtained from (5.56) by a representation formula argument as before. \square

Proposition 5.1 provides more information than needed to just prove Proposition 5.3. The precision of estimates (5.1) and (5.2) will turn out to be crucial in section 7 to obtain precise asymptotic expansions of the λ_k^i .

It is worth noting that Propositions 5.1 and 5.3 do depend on the choice of X given by (2.24) (mostly to obtain pointwise estimates on the source term) but do not use the specific form of the other coefficients, e.g. of h and f defined in (2.7), (2.10) and (2.23). In particular, Propositions 5.1 and 5.3 still hold true for arbitrary choices of $h, \pi \in C^0(M)$, $f \in C^1(M)$, $\sigma \in C^0(M)$ and $Y \in C^0(M)$.

6. GLOBAL FIXED-POINT ARGUMENT AND RESOLUTION OF THE REDUCED PROBLEM

Proposition 5.3 shows that we have pointwise estimates on the remainder ϕ_k defined by Proposition 3.3 which only depend on the data μ_k and ε_k , and not on the chosen sequences $(t_k, \xi_k)_k$ or $(v_k)_k$.

In this section we crucially use this result to show that Banach-Picard's fixed-point theorem applies to the mapping $v_k \mapsto \phi_k$ and yields a solution to the reduced problem for system (1.1). The main result of this section is the following:

Proposition 6.1. *Let $D > 0$. Assume that η and α defined in (2.11) and (2.25) are small enough. There exists $k_0 \in \mathbb{N}$ such that for any sequence $(t_k, \xi_k)_k \in [1/D, D] \times M$ and for any $k \geq k_0$, there*

exists a function $\phi_k = \phi_k(t_k, \xi_k) \in K_{k, t_k, \xi_k}^\perp$ that satisfies the following system of equations:

$$\begin{cases} \Pi_{K_{k, t_k, \xi_k}^\perp} \left[u_k - (\Delta_g + h)^{-1} \left(f u_k^{2^*-1} + \frac{|\mathcal{L}_g T_k + \sigma|_g^2 + \pi^2}{u_k^{2^*+1}} \right) \right] = 0, \\ \vec{\Delta}_g T_k = u_k^{2^*} X + Y, \end{cases} \quad (6.1)$$

where, as in (4.2), we have let $u_k = u + W_{k, t_k, \xi_k} + \phi_k(t_k, \xi_k)$. In addition, there exists a positive constant C , independent of $(t_k, \xi_k)_k$ such that there holds:

$$\|\phi_k(t_k, \xi_k)\|_{H^1(M)} \leq C\delta_k \text{ and } |\phi_k(t_k, \xi_k)| \leq C\delta_k(u + W_{k, t_k, \xi_k}) \text{ in } M, \quad (6.2)$$

and such that $\phi_k(t_k, \xi_k)$ is the unique solution of (6.1) in K_{k, t_k, ξ_k}^\perp satisfying in addition (6.2). Also, for any k , the mapping $(t, \xi) \mapsto \phi_k(t, \xi) \in C^1(M)$ is continuous.

In Proposition 6.1, K_{k, t_k, ξ_k}^\perp is again as in (2.22). The smallness assumption on η and α is made clear in the course of the proof.

Proof. We let $C_0 = C_0(n, g, u_0, h, f, \sigma, \pi, D)$ denote the smallest of the two positive constants appearing in equations (5.56) and (3.9), and we let:

$$\varepsilon_k = 4C_0\delta_k. \quad (6.3)$$

Assume that η in (2.11) is chosen so that there holds:

$$C_0\eta \leq \frac{1}{4}. \quad (6.4)$$

Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$ and let $F_k = F(\varepsilon_k, t_k, \xi_k)$, where $F(\varepsilon_k, t_k, \xi_k)$ is defined in (2.27) and ε_k is as in (6.3). For any k , we define the following mapping:

$$\Psi_k : \begin{cases} F(\varepsilon_k, t_k, \xi_k) \rightarrow F(\varepsilon_k, t_k, \xi_k) \cap \left\{ \varphi \in K_{k, t_k, \xi_k}^\perp, \|\varphi\|_{H^1(M)} \leq \frac{\varepsilon_k}{4} \right\}, \\ v \mapsto \phi_k = \phi_k(t_k, \xi_k, v) \end{cases} \quad (6.5)$$

where $\phi_k(t_k, \xi_k, v)$ is given by Proposition 3.3. For any k , we endow the set F_k with the norm

$$\|v\|_{F_k} = \left\| \frac{v}{u + W_{k, t_k, \xi_k}} \right\|_{C^0(M)}. \quad (6.6)$$

That the mapping Ψ_k in (6.5) is well-defined for $k \geq k_0$ is a consequence of (6.3), (6.4), Proposition 5.3 and (3.9), which also show that the value of such a k_0 is independent of the choice of the sequence $(t_k, \xi_k)_k$. We now show that for k sufficiently large the mapping Ψ_k is a contraction for the norm (6.6).

Let, for any k , $v_k^1, v_k^2 \in F_k$ and denote by ϕ_k^1 and ϕ_k^2 the associated images by Ψ_k . Using coherent notations, for $i = 1, 2$, we will let $u_k^i = u + W_{k, t_k, \xi_k} + \phi_k^i$ and T_k^i will denote the unique solution of $\vec{\Delta}_g T_k^i = u_k^{i, 2^*} X + Y$ in M . As before, we shall omit the dependence in t_k and ξ_k in the computations.

By Proposition 3.3, for any k there exist $(\lambda_{k,j}^1)_{0 \leq j \leq n}$ and $(\lambda_{k,j}^2)_{0 \leq j \leq n}$ such that $\phi_k^1 - \phi_k^2$ satisfies:

$$\begin{aligned} (\Delta_g + h)(\phi_k^1 - \phi_k^2) &= f \left[\left(u + W_k + \phi_k^1 \right)^{2^*-1} - \left(u + W_k + \phi_k^2 \right)^{2^*-1} \right] \\ &\quad + \left(\pi^2 + |\sigma + \mathcal{L}_g T_g|^2 \right) \left[\left(u + W_k + \phi_k^1 \right)^{-2^*-1} - \left(u + W_k + \phi_k^2 \right)^{-2^*-1} \right] \\ &\quad + \left(|\sigma + \mathcal{L}_g T_k^1|_g^2 - |\sigma + \mathcal{L}_g T_g|^2 \right) \left[\left(u + W_k + v_k^1 \right)^{-2^*-1} - \left(u + W_k + v_k^2 \right)^{-2^*-1} \right] \\ &\quad + \left(u + W_k + v_k^2 \right)^{-2^*-1} \left(|\sigma + \mathcal{L}_g T_k^1|_g^2 - |\sigma + \mathcal{L}_g T_g|^2 \right) \\ &\quad + \sum_{j=0}^n (\lambda_{k,j}^1 - \lambda_{k,j}^2) (\Delta_g + h) Z_{j,k}. \end{aligned} \quad (6.7)$$

We first estimate the H^1 -norm of $\phi_k^1 - \phi_k^2$. Since $v_k^i \in F_k$ there holds, with (2.11):

$$|\vec{\Delta}_g(T_k^1 - T_k^2)| \lesssim (u + W_k)^{2^*} \eta \|v_k^1 - v_k^2\|_{F_k},$$

so that mimicking the proof of Proposition 9.2 yields, with (2.24): for any $x \in M$,

$$|\mathcal{L}_g(T_k^1 - T_k^2)|(x) \lesssim \eta (\delta_k^{\frac{n-1}{2}} \theta_k(x)^{1-n} + 1) \|v_k^1 - v_k^2\|_{F_k}. \quad (6.8)$$

We now apply $(\Delta_g + h)^{-1}$ to (6.7) and project on $K_k^\perp = K_{k,t_k,\xi_k}^\perp$. Using (9.18), Proposition 3.5 and the techniques developed in the proof of Proposition 3.3 we then get that:

$$\|\phi_k^1 - \phi_k^2\|_{H^1(M)} \lesssim \eta \|v_k^1 - v_k^2\|_{F_k}. \quad (6.9)$$

Let now $0 \leq j \leq n$ and integrate (6.7) against $Z_{j,k}$. With (6.9) one obtains:

$$\sum_{j=0}^n |\lambda_{k,j}^1 - \lambda_{k,j}^2| \lesssim \eta \|v_k^1 - v_k^2\|_{F_k}. \quad (6.10)$$

We now show that, up to suitably choosing η and α as in (2.11) and (2.25), for k large enough there always holds:

$$\|\phi_k^1 - \phi_k^2\|_{F_k} \leq \frac{1}{2} \|v_k^1 - v_k^2\|_{F_k}. \quad (6.11)$$

For this, we let $\Phi_k = \phi_k^1 - \phi_k^2$ and let x_k be the point where the F_k -norm of Φ_k is attained:

$$\left| \frac{\Phi_k}{u + W_k}(x_k) \right| = \sup_{x \in M} \left| \frac{\Phi_k}{u + W_k}(x) \right| = \|\Phi_k\|_{F_k}. \quad (6.12)$$

We distinguish between two cases. We first assume that $d_{g_{\xi_k}}(\xi_k, x_k) = O(\delta_k)$. We then let, for any $x \in B_0(i_g(M)/\delta_k)$:

$$\tilde{\Phi}_k(x) = \frac{\delta_k^{\frac{n-2}{2}} \Phi_k}{\|\Phi_k\|_{F_k} + \|v_k^1 - v_k^2\|_{F_k}} (\exp^{g_{\xi_k}}(\delta_k x)). \quad (6.13)$$

We also let $\tilde{x}_k = \frac{1}{\delta_k} \exp^{g_{\xi_k}}(x_k)$. There holds then: $\tilde{x}_k \rightarrow \tilde{x}_0 \in \mathbb{R}^n$ as $k \rightarrow +\infty$ with $|\tilde{x}_0| = R_0$ for some $R_0 \geq 0$. It is easily seen that $\|\tilde{\Phi}_k\|_{L^\infty} \leq 1$ and by (6.7), using Proposition 5.3 and standard elliptic theory, we get that the sequence $\tilde{\Phi}_k$ converges in $C_{loc}^1(\mathbb{R}^n)$ to some function $\tilde{\Phi}_0$ which is a solution of:

$$\Delta_\xi \tilde{\Phi}_0 = (2^* - 1) f(\xi_0) U_{\xi_0}^{2^*-2} + \sum_{j=0}^n \tilde{\lambda}_{0,j} \Delta_\xi V_{j,\xi_0}, \quad (6.14)$$

and which satisfies, for any $x \in \mathbb{R}^n$:

$$|\tilde{\Phi}_0(x)| \leq |\tilde{\Phi}_0(\tilde{x}_0)| = \left(\lim_{k \rightarrow +\infty} \frac{\|\Phi_k\|_{F_k}}{\|\Phi_k\|_{F_k} + \|v_k^1 - v_k^2\|_{F_k}} \right) \left(1 + \frac{f(\xi_0)}{n(n-2)} R_0^2 \right)^{1-\frac{n}{2}}. \quad (6.15)$$

In (6.14) U_{ξ_0} and the V_{j,ξ_0} are defined in (2.20) and (2.18), and we have let

$$\tilde{\lambda}_{0,j} = \lim_{k \rightarrow +\infty} \frac{\lambda_{k,j}^1 - \lambda_{k,j}^2}{\|\Phi_k\|_{F_k} + \|v_k^1 - v_k^2\|_{F_k}}.$$

This limit exists, up to a subsequence, by (6.10). A first thing to notice is that $\tilde{\Phi}_0 \in L^{2^*}(\mathbb{R}^n)$. This is a consequence of (6.9), of the scaling invariance of the L^{2^*} norm and of the definition of $\tilde{\Phi}_k$ in (6.13). Therefore, we can integrate (6.14) against V_{j,ξ_0} for all $0 \leq j \leq n$ and (2.19) shows that there holds $\tilde{\lambda}_{0,j} = 0$ for all $0 \leq j \leq n$. Since $\tilde{\Phi}_0 \in L^{2^*}(\mathbb{R}^n)$, there holds then $\tilde{\Phi}_0 \in H^1(\mathbb{R}^n)$ and the Bianchi-Egnell [5] classification result applies and shows that

$$\tilde{\Phi}_0 \in \text{Span}\{V_{j,\xi_0}, 0 \leq j \leq n\}. \quad (6.16)$$

Similarly to what we did in the proof of Claim 5.2, we will now show that $\tilde{\Phi}_0 \in \text{Span}\{V_{j,\xi_0}, 0 \leq j \leq n\}^\perp$. Since $\Phi_k \in K_k^\perp$ we can write, for any $R > 0$ and $0 \leq j \leq n$, that

$$\begin{aligned} \int_{B_{\xi_k}(R\delta_k)} \langle \nabla Z_{j,k}, \nabla \Phi_k \rangle_g + h Z_{j,k} \Phi_k dv_g &= - \int_{\partial B_{\xi_k}(R\delta_k)} \Phi_k \partial_\nu Z_{j,k} d\sigma_g \\ &- \int_{M \setminus B_{\xi_k}(R\delta_k)} (h - c_n S_g) Z_{j,k} \Phi_k dv_g - \int_{M \setminus B_{\xi_k}(R\delta_k)} (\triangle_g + c_n S_g) Z_{j,k} \Phi_k dv_g. \end{aligned} \quad (6.17)$$

Mimicking the computations that led to (5.29) – (5.37) and using (9.2) and the definition of the $\|\cdot\|_{F_k}$ -norm in (6.6) one obtains that, for any $0 \leq j \leq n$:

$$\left| \int_{B_{\xi_k}(R\delta_k)} \langle \nabla Z_{j,k}, \nabla \Phi_k \rangle_g + h Z_{j,k} \Phi_k dv_g \right| \lesssim \left(o(1) + R^{-2} \right) \|\Phi_k\|_{F_k}. \quad (6.18)$$

Dividing both sides of (6.18) by $\|\Phi_k\|_{F_k} + \|v_k^1 - v_k^2\|_{F_k}$, using the definition of $\tilde{\Phi}_k$ in (6.13) and the convergence of $\tilde{\Phi}_k$ to $\tilde{\Phi}_0$, letting $k \rightarrow +\infty$ and then $R \rightarrow +\infty$, we then obtain that $\tilde{\Phi}_0 \in \text{Span}\{V_{j,\xi_0}, 0 \leq j \leq n\}^\perp$. With (6.16), this gives that $\tilde{\Phi}_0 \equiv 0$. Using (6.15) this gives in turn that

$$\|\phi_k^1 - \phi_k^2\|_{F_k} = o\left(\|v_k^1 - v_k^2\|_{F_k}\right),$$

which proves (6.11) in this case.

We now assume that there holds, up to a subsequence, that

$$\frac{d_{g_{\xi_k}}(\xi_k, x_k)}{\delta_k} \rightarrow +\infty \quad (6.19)$$

as $k \rightarrow +\infty$, where x_k is given by (6.12). Recall the definition of the operator L_u introduced in (5.58). Then, using (5.56) and (6.3), we rewrite (6.7) as:

$$\begin{aligned} L_u(\Phi_k) &= O\left(\min(W_k^{2^*-2}, W_k)|\Phi_k|\right) + o(|\Phi_k|) \\ &\quad + \left(|\sigma + \mathcal{L}_g T_k^1|_g^2 - |\sigma + \mathcal{L}_g T|_g^2\right) \left[\left(u + W_k + v_k^1\right)^{-2^*-1} - \left(u + W_k + v_k^2\right)^{-2^*-1}\right] \\ &\quad + \left(u + W_k + v_k^2\right)^{-2^*-1} \left(|\sigma + \mathcal{L}_g T_k^1|_g^2 - |\sigma + \mathcal{L}_g T_k^2|_g^2\right) \\ &\quad + \sum_{j=0}^n (\lambda_{k,j}^1 - \lambda_{k,j}^2) (\triangle_g + h) Z_{j,k}. \end{aligned} \quad (6.20)$$

By (2.13), the Green's function G_u of L_u satisfies the pointwise bounds (5.60). We now write a Green representation formula for L_u at x_k . Using (6.20), (6.8), (6.10), (9.11) below together with (2.24) and (5.60) we get that there holds:

$$\begin{aligned} |\Phi_k(x_k)| &\leq o\left(\|\Phi_k\|_{F_k} \left(u + W_k\right)(x_k)\right) + D_0(\alpha + \eta) \|v_k^1 - v_k^2\|_{F_k} \\ &\quad + \int_M G_u(x_k, y) \min\left(W_k^{2^*-2}(y), W_k(y)\right) |\Phi_k|(y) dv_g(y) \end{aligned} \quad (6.21)$$

for some positive D_0 that does not depend on k , and where η and α are as in (2.11) and (2.25). Let $R > 0$ be fixed. We have, using (6.19), that:

$$\begin{aligned} &\left| \int_{M \setminus B_{\xi_k}(R\delta_k)} G_u(x_k, y) \min\left(W_k^{2^*-2}(y), W_k(y)\right) |\Phi_k|(y) dv_g(y) \right| \\ &\lesssim \int_{M \setminus B_{\xi_k}(\sqrt{\delta_k})} G_u(x_k, y) W_k(y) \|\Phi_k\|_{F_k} dv_g(y) \\ &\quad + \int_{B_{\xi_k}(\sqrt{\delta_k}) \setminus B_{\xi_k}(R\delta_k)} G_u(x_k, y) W_k^{2^*-1}(y) \|\Phi_k\|_{F_k} dv_g(y) \\ &\leq D_1 \left(\frac{1}{R^2} + \delta_k\right) \left(u + W_k\right)(x_k) \|\Phi_k\|_{F_k}, \end{aligned} \quad (6.22)$$

for some positive constant D_1 which does not depend on k or on R . Independently, if we let $p > \frac{n}{4}$ be fixed, two Hölder inequalities together with (6.19) and (6.9) show that

$$\int_{B_{\xi_k}(R\delta_k)} G_u(x_k, y) W_k^{2^*-2} |\Phi_k|(y) dv_g(y) \lesssim D_p R^{\frac{n+2}{2} - \frac{n}{p}} \eta \|v_k^1 - v_k^2\|_{F_k} \delta_k^{\frac{n-2}{2}} d_{g_{\xi_k}}(\xi_k, x_k)^{2-n}, \quad (6.23)$$

for some positive constant D_p which depends on p but not on k or R . Choose now $R_0 > 0$ such that $D_1 R_0^{-2} = \frac{1}{8}$, where D_1 is given by (6.22). Assume then that α and η are small enough to have $D_p R_0^{\frac{n+2}{2} - \frac{n}{p}} \eta \leq \frac{1}{8}$ and $D_0(\alpha + \eta) \leq \frac{1}{8}$, where D_0 and D_p are given by (6.21) and (6.23). Then, plugging (6.22) and (6.23) in (6.21) and using the definition (6.12) of x_k gives:

$$\|\phi_k^1 - \phi_k^2\|_{F_k} \leq \frac{1}{2} \|v_k^1 - v_k^2\|_{F_k},$$

thus concluding the proof of (6.11). The fact that (6.11) holds for large k independent on the choice of $(t_k, \xi_k)_k$ follows by a standard contradiction argument, up to passing to a subsequence.

Now, the uniqueness property that defines $\phi(t_k, \xi_k, v_k)$ (and is stated in Proposition 3.3) shows that, for any k large enough, a function $\varphi \in F(\varepsilon_k, t_k, \xi_k) \cap \{\varphi \in K_{k,t_k,\xi_k}^\perp, \|\varphi\|_{H^1(M)} \leq \frac{\varepsilon_k}{4}\}$ solves (6.1)

if and only if it is a fixed-point of Ψ_k defined in (6.5). Using (6.11), Banach-Picard's fixed-point theorem asserts, for any k , the existence of such a fixed-point $\phi_k(t_k, \xi_k)$ as well as its uniqueness in $F(C\delta_k, t_k, \xi_k) \cap K_{k,t_k,\xi_k}^\perp \cap B_{H^1(M)}(0, C\delta_k)$. The estimates in (6.2) follow then from (3.9), (5.3) and (6.3). Finally, the continuity of the mapping $(t, \xi) \in (0, +\infty) \times M \mapsto \phi_k(t, \xi) \in C^1(M)$ follows from direct arguments, using (4.5), standard elliptic theory and the uniqueness property of $\phi_k(t, \xi)$ in $F(C\delta_k, t_k, \xi_k) \cap K_{k,t_k,\xi_k}^\perp \cap B_{H^1(M)}(0, C\delta_k)$, thus concluding the proof of Proposition 6.1. \square

One might be surprised by the use of Banach-Picard's fixed-point theorem in the proof of Proposition 6.1. In view of Proposition 5.3, an application of Schauder's fixed-point theorem would seem preferable to construct a solution of (6.1) – and the proof would indeed be simpler. The reason for using Banach-Picard's theorem is that Schauder's theorem does not provide a preferred solution of (6.1), and in particular yields no uniqueness property on the remainder $\phi_k(t, \xi)$ thus constructed. A striking consequence is that if $\phi_k(t, \xi)$ is not uniquely determined in some way, one is not able to prove its continuity in the choice of $(t, \xi) \in (0, +\infty) \times M$.

This remark is best understood anticipating a little on Sections 7 and 8. If we were to construct a remainder $\phi_k(t, \xi)$ by Schauder's fixed-point theorem it would still satisfy (6.2) and the (analogue of the) estimates of Proposition 5.1, so that the expansions we perform in Section 7 would still hold true. In particular, one would obtain an exact analogue of (8.2), but where the error terms $R_k^i(t, p)$, $0 \leq i \leq n$, would only be bounded functions whose uniform bound goes to zero as $k \rightarrow +\infty$. Since $(t, \xi) \mapsto \phi_k(t, \xi)$ is not continuous, these $R_k^i(t, p)$ would be non-continuous too, and the usual final annihilating arguments would fail.

The pointwise estimates obtained in Proposition 5.3 still hold true for the function $\phi_k(t, \xi)$ given by Proposition 6.1, when substituting ε_k by μ_k . Estimate (9.11) also remains true for the field of 1-forms $T_{k,t,\xi}$ associated to $u + W_{k,t,\xi} + \phi_k(t, \xi)$ by (3.3). Proposition 5.1 gives the following control:

Proposition 6.2. *Let $D > 0$, $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, and let $\phi_k = \phi_k(t_k, \xi_k)$ denote the function given by Proposition 6.1. Let $(x_k)_k$ be any sequence of points in $B_{\xi_k}(2\sqrt{\delta_k})$. Then there holds:*

- If $n \geq 7$:

$$\begin{aligned} |\phi_k(x_k)| &\lesssim \delta_k + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \delta_k \|h - c_n S_g\|_{L^\infty(2r_k)} + \delta_k \\ &\quad + \left(\frac{\delta_k}{\theta_k(x_k)} \right)^2 + \left[\delta_k^{\frac{n}{2}} + \delta_k \|\nabla f\|_{L^\infty(2r_k)} \right. \\ &\quad \left. + \|h - c_n S_g\|_{L^\infty(2r_k)} \left(\theta_k(x_k)^2 + \delta_k^2 \ln \left(\frac{\theta_k(x_k)}{\delta_k} \right) \right) + \theta_k(x_k)^4 \mathbb{1}_{n \leq 7} \right] W_k(x_k). \end{aligned}$$

- If $n = 6$:

$$\begin{aligned} |\phi_k(x_k)| &\lesssim \delta_k + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \delta_k \left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)} \\ &\quad + \left[\delta_k^3 + \delta_k \|\nabla f\|_{L^\infty(2r_k)} + \left\| h - \frac{1}{5} S_g - 2fu \right\|_{L^\infty(2r_k)} \left(\theta_k(x_k)^2 + \delta_k^2 \ln \left(\frac{\theta_k(x_k)}{\delta_k} \right) \right) \right] W_k(x_k). \end{aligned}$$

Proof. Proposition 5.3 shows that there holds:

$$\|\phi_k\|_{L^\infty(B_{\xi_k}(2r_k) \setminus B_{\xi_k}(\sqrt{\delta_k}))} + \sqrt{\delta_k} \|\nabla \phi_k\|_{L^\infty(B_{\xi_k}(2r_k) \setminus B_{\xi_k}(\sqrt{\delta_k}))} \lesssim \delta_k,$$

and the result then follows from an application of Proposition 5.1. \square

To be able to perform the asymptotic expansion along $K_{k,t,\xi}^\perp$ in the next section, we will need more precise estimates than (5.56) on the behaviour of $\phi_k(t, \xi)$ at distances from ξ_k which are large compared to $\sqrt{\mu_k}$. We quantify more precisely the fall-off of $\phi_k(t, \xi)$ far away from the center of the bubbles in the next Proposition:

Proposition 6.3. *Let $D > 0$, $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, and let $\phi_k = \phi_k(t_k, \xi_k)$ denote the function given by Proposition 6.1. Let $(R_k)_k$, $R_k \geq 1$, denote a sequence of positive numbers. There holds:*

$$\|\phi_k\|_{L^\infty(M \setminus B_{\xi_k}(R_k \sqrt{\delta_k}))} \lesssim \frac{\delta_k}{R_k^2} + R_k^2 \delta_k^2 + \delta_k^{\frac{n-2}{2}} r_k^{-n}. \quad (6.24)$$

Proof. Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$ and let $(R_k)_k$, $R_k \geq 1$, denote a sequence of positive numbers. Let $(x_k)_k$ be a sequence of points in M satisfying $d_{g_{\xi_k}}(\xi_k, x_k) \geq R_k \sqrt{\delta_k}$. Let again G_u denote the Green's function of the operator L_u defined in (5.58). We use (5.68) and (9.18) below to write that there holds, for some positive constant C that neither depends on k nor on η as in (2.11), that:

$$\begin{aligned} \int_M G_u(x_k, y) \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{(u + W_k + v_k)^{2^*+1}}(y) dv_g(y) &\leq C \left(\delta_k^{\frac{n}{2}} \theta_k(x_k)^{2-n} \right. \\ &\quad \left. + \eta \|\phi_k\|_{L^\infty(M \setminus B_{\xi_k}(R_k \sqrt{\delta_k}))} + \delta_k^2 + \eta R_k^2 \delta_k^2 + \delta_k^{\frac{n-1}{2}} \theta_k(x_k)^{3-n} \right). \end{aligned} \quad (6.25)$$

Here, $T_k = T_{k,t_k,\xi_k}$ is the solution of the 1-form equation in (6.1). Assume now that η in (2.11) is small enough so as to have

$$C\eta \leq \frac{1}{2},$$

where C is the constant appearing in (6.25). Estimate (6.24) then follows from (6.25) by choosing the sequence $(x_k)_k$ to be such that

$$|\phi_k(x_k)| = \|\phi_k\|_{L^\infty(M \setminus B_{\xi_k}(R_k \sqrt{\delta_k}))}$$

and by writing a representation formula for L_u at x_k . The representation formula is written for (5.59) and the terms appearing in it are estimated by using (5.61)–(5.67) and (6.2). \square

The considerations in the remark following Proposition 5.3 apply here too: Propositions 6.1, 6.2 and 6.3 still hold true for arbitrary choices of $h, \pi \in C^0(M)$, $f \in C^1(M)$, $\sigma \in C^0(M)$ and $Y \in C^0(M)$ once X is given by (2.24).

7. EXPANSION OF THE KERNEL COEFFICIENTS

Let $D > 0$. We let, for any $p \in \overline{B_0(1)}$:

$$y_k = \exp_{\xi_k}^{g_{\xi_k}}(\beta_k p), \quad (7.1)$$

where β_k is defined in (2.6) and $(\xi_k)_k$ is the sequence chosen in Section 2. Throughout this section the functions W_{k,t,y_k} and Z_{i,k,t,y_k} defined in (2.17) and (2.21) will be denoted by $W_{k,t,p}$ and $Z_{i,k,t,p}$, with $\delta_k(t)$ again given by (2.16). For any $(t, p) \in [1/D, D] \times \overline{B_0(1)}$ we will denote by $\phi_k(t, p)$ the function $\phi_k(t, y_k)$ given by Proposition 6.1 when y_k is given by (7.1), and as before we let $u_{k,t,p} = u + W_{k,t,p} + \phi_k(t, p)$. Proposition 6.1 shows that there exist real numbers $(\lambda_k^i(t, p))_{0 \leq i \leq n}$ such that $u_{k,t,p}$ satisfies:

$$\begin{cases} (\Delta_g + h) u_{k,t,p} = f u_{k,t,p}^{2^*-1} + \frac{|\mathcal{L}_g T_{k,t,p} + \sigma|_g^2 + \pi^2}{u_{k,t,p}^{2^*+1}} + \sum_{i=0}^n \lambda_k^i(t, p) (\Delta_g + h) Z_{i,k,t,p}, \\ \vec{\Delta}_g T_{k,t,p} = u_{k,t,p}^{2^*} X + Y. \end{cases} \quad (7.2)$$

Since $\phi_k(t, p)$, $W_{k,t,p}$ and $Z_{i,k,t,p}$ are continuous in the choice of (t, p) , then so are the $(\lambda_k^i(t, p))_{0 \leq i \leq n}$.

In this section we conclude the proof of Theorem 1.1 by showing that for any k there exists (t_k, p_k) such that $\lambda_k^i(t_k, p_k) = 0$ for any $0 \leq i \leq n$. By (7.2), the function u_{k,t_k,p_k} will therefore provide the desired solution of (1.1) and this will conclude the proof of Theorem 1.1. We do this by performing an asymptotic expansion of the $(\lambda_k^i(t, p))_{0 \leq i \leq n}$ as $k \rightarrow +\infty$ towards some limiting function that possesses zeroes. As before, we distinguish between the $n = 6$ and $n \geq 7$ cases. All the expansions obtained in this section will be uniform in the choice of $(t, p) \in [1/D, D] \times \overline{B_0(1)}$. In this section we will often apply previously obtained results, such as Proposition 6.1. They will always be applied for the sequence (t, y_k) where y_k is given by (7.1). As mentioned in the Introduction, the expansions in this section do not rely on the assumption $r_k \rightarrow 0$.

7.1. The $n \geq 7$ case. We start by re-writing the scalar equation in (7.2) as:

$$\begin{aligned}
& \sum_{i=0}^n \lambda_k^i(t, p) (\Delta_g + h) Z_{i,k,t,p} = \\
& (\Delta_g + h) W_{k,t,p} - f(y_k) W_{k,t,p}^{2^*-1} + (f(y_k) - f) W_{k,t,p}^{2^*-1} \\
& - f \left[(u + W_{k,t,p} + \phi_k(t, p))^{2^*-1} - (u + W_{k,t,p})^{2^*-1} - (2^* - 1)(u + W_{k,t,p})^{2^*-2} \phi_k(t, p) \right] \\
& - f \left[(u + W_{k,t,p})^{2^*-1} - u^{2^*-1} - W_{k,t,p}^{2^*-1} \right] \\
& + (\Delta_g + h) \phi_k(t, p) - (2^* - 1) f(y_k) W_{k,t,p}^{2^*-2} \phi_k(t, p) \\
& - (2^* - 1) f \left[(u + W_{k,t,p})^{2^*-2} - W_{k,t,p}^{2^*-2} \right] \phi_k(t, p) \\
& + (2^* - 1) (f(y_k) - f) W_{k,t,p}^{2^*-2} \phi_k(t, p) \\
& + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} - \frac{|\mathcal{L}_g T_{k,t,p} + \sigma|_g^2 + \pi^2}{(u + W_{k,t,p} + \phi_k(t, p))^{2^*+1}}.
\end{aligned} \tag{7.3}$$

We integrate each side of equation (7.3) against $Z_{i,k,t,p}$, for a given $0 \leq i \leq n$. For any $0 \leq i \leq n$ we let:

$$\begin{aligned}
I_{1,i} &= \int_M \left((\Delta_g + h) W_{k,t,p} - f(y_k) W_{k,t,p}^{2^*-1} + (f(y_k) - f) W_{k,t,p}^{2^*-1} \right) Z_{i,k,t,p} dv_g, \\
I_{2,i} &= - \int_M f \left[(u + W_{k,t,p} + \phi_k(t,p))^{2^*-1} - (u + W_{k,t,p})^{2^*-1} \right. \\
&\quad \left. - (2^* - 1)(u + W_{k,t,p})^{2^*-2} \phi_k(t,p) \right] Z_{i,k,t,p} dv_g, \\
I_{3,i} &= - \int_M f \left[(u + W_{k,t,p})^{2^*-1} - u^{2^*-1} - W_{k,t,p}^{2^*-1} \right] Z_{i,k,t,p} dv_g, \\
I_{4,i} &= \int_M \left[(\Delta_g + h) \phi_k(t,\xi) - (2^* - 1) f(\xi_k) W_{k,t,\xi}^{2^*-2} \phi_k(t,\xi) \right] Z_{i,k,t,\xi} dv_g, \\
I_{5,i} &= -(2^* - 1) \int_M f \left[(u + W_{k,t,p})^{2^*-2} - W_{k,t,p}^{2^*-2} \right] \phi_k(t,p) Z_{i,k,t,p} dv_g, \\
I_{6,i} &= (2^* - 1) \int_M (f(y_k) - f) W_{k,t,p}^{2^*-2} \phi_k(t,p) Z_{i,k,t,p} dv_g, \\
I_{7,i} &= \int_M \left(\frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} - \frac{|\mathcal{L}_g T_{k,t,p} + \sigma|_g^2 + \pi^2}{(u + W_{k,t,p} + \phi_k(t,p))^{2^*+1}} \right) Z_{i,k,t,p} dv_g.
\end{aligned} \tag{7.4}$$

We will need precise asymptotic expansions of the $\lambda_k^i(t,p)$ and so will always distinguish in the following between the $i = 0$ and $1 \leq i \leq n$ cases, to take into account the different decay of the $Z_{i,k,t,p}$. Our aim is to show that the dominant contributions in the expansion of the $\lambda_k^i(t,p)$ come from the integrals $I_{1,i}$ and $I_{3,i}$ in (7.4).

The integrals in (7.4) are computed in a series of Claims. We start with the computations of $I_{1,i}$:

Claim 7.1. *There holds:*

$$\begin{aligned}
I_{1,0} &= \frac{8(n-1)}{(n-2)(n-4)} K_n^{-n} f(\xi_0)^{-\frac{n}{2}} \tau_k \mu_k^2 H(p) t^2 \\
&\quad - \frac{1}{3} \frac{n(n-2)}{(n-4)(n-6)} f(\xi_0)^{-1-\frac{n}{2}} K_n^{-n} |W_g(\xi_0)|_g^2 \mu_k^4 t^4 + o(\tau_k \delta_k^2) + o(\delta_k^4 \mathbf{1}_{nlcf}) + o(\delta_k^{\frac{n-2}{2}}),
\end{aligned} \tag{7.5}$$

and, for any $1 \leq i \leq n$:

$$\begin{aligned}
I_{1,i} &= \frac{2n(n-1)}{n-4} K_n^{-n} f(\xi_0)^{-\frac{n}{2}} \frac{\tau_k}{\beta_k} \mu_k^3 \nabla_i H(p) t^3 \\
&\quad - f(\xi_0)^{-1-\frac{n}{2}} K_n^{-n} \frac{n^2(n-2)^2}{24(n-4)(n-6)} \nabla_i (|W(\cdot)|_g^2)(\xi_0) \mu_k^5 t^5 + o(\delta_k^5 \mathbf{1}_{nlcf}) + o(\delta_k^{\frac{n}{2}}).
\end{aligned} \tag{7.6}$$

Here $\xi_0 = \lim_{k \rightarrow +\infty} \xi_k$ and τ_k and μ_k are as in (2.5).

In (7.5) and (7.6) $W_g(\xi_0)$ denotes the Weyl tensor at ξ_0 (which vanishes if (M, g) is locally conformally flat) and K_n^{-n} is the H^1 -energy of the standard bubble defined by:

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{\frac{2}{n}}}}, \tag{7.7}$$

where ω_n is the volume of the standard unit n -sphere.

Proof. Using the definition of conformal normal coordinates as in (2.1) and (2.3), and using (2.17) and (2.21), there holds, for any fixed $t > 0$ and $\xi \in M$, that:

$$\begin{aligned} \frac{\partial}{\partial t} W_{k,\delta,\xi} &= \frac{n-2}{2t} Z_{0,k,t,\xi}, \\ \frac{\partial}{\partial \xi_i} W_{k,t,\xi} &= \frac{f(\xi)}{n\delta_k(t)} Z_{i,k,t,\xi} + O(\delta_k^{\frac{n-2}{2}} r_k^{1-n} \mathbf{1}_{r_k \leq d_{g_\xi} \leq 2r_k}) + O(|\nabla f(\xi)|_g W_{k,t,\xi}) + O(d_{g_\xi}(\xi, \cdot) W_{k,t,\xi}). \end{aligned}$$

These expressions are for instance obtained from the estimations in Appendix A in Esposito-Pistoia-Vétois [23] and those in Section 4 of Robert-Vétois [45]. Estimates (7.5) and (7.6) follow then from straightforward computations using (2.7), (2.23) and (2.15). \square

Claim 7.2. *There holds:*

$$I_{2,0} = o(\delta_k^{\frac{n-2}{2}}) \text{ if } (M, g) \text{ is l.c.f. or if } n \leq 10, \quad (7.8)$$

$$I_{2,0} = o(\delta_k^4) \text{ if } (M, g) \text{ is not l.c.f. and } n \geq 11, \quad (7.9)$$

and, for $1 \leq i \leq n$,

$$I_{2,i} = o(\delta_k^{\frac{n}{2}}) + o(\delta_k^5 \mathbf{1}_{n\text{l.c.f.}}). \quad (7.10)$$

Proof. First, we have that:

$$\begin{aligned} f \left[(u + W_{k,t,p} + \phi_k(t,p))^{2^*-1} - (u + W_{k,t,p})^{2^*-1} - (2^* - 1)(u + W_{k,t,p})^{2^*-2} \phi_k(t,p) \right] \\ \lesssim (u + W_{k,t,p})^{2^*-3} |\phi_k(t,p)|^2. \end{aligned}$$

Using (6.2) and since $n \geq 7$ and $|Z_{0,k,t,p}| \lesssim W_{k,t,p}$, we write that:

$$\begin{aligned} \int_{M \setminus B_{y_k}(\sqrt{\delta_k})} (u + W_{k,t,p})^{2^*-3} |\phi_k(t,p)|^2 |Z_{0,k,t,p}| dv_g \lesssim \delta_k^2 \int_{B_{y_k}(2r_k) \setminus B_{y_k}(\sqrt{\delta_k})} |Z_{0,k,t,p}| dv_g \\ + \delta_k^2 \int_{B_{y_k}(2r_k) \setminus B_{y_k}(\sqrt{\delta_k})} W_{k,t,p}^{2^*} dv_g \end{aligned}$$

which gives then

$$\int_{M \setminus B_{y_k}(\sqrt{\delta_k})} (u + W_{k,t,p})^{2^*-3} |\phi_k(t,p)|^2 |Z_{0,k,t,p}| dv_g = o(\delta_k^{\frac{n}{2}}). \quad (7.11)$$

We now use Proposition 6.2 and (2.23) to write that there holds:

$$\int_{B_{y_k}(\sqrt{\delta_k})} W_{k,t,p}^{2^*-3} |\phi_k(t,p)|^2 |Z_{0,k,t,p}| dv_g \lesssim o(\delta_k^{\frac{n}{2}}) + o(\delta_k^4 \mathbf{1}_{n\text{l.c.f.}}) + \delta_k^4 \|h - c_n S_g\|_{L^\infty(2r_k)}^2. \quad (7.12)$$

Assume first that (M, g) is locally conformally flat or that $n \leq 10$. Then, using (2.7) there holds $\|h - c_n S_g\|_{L^\infty(2r_k)}^2 = \delta_k^{n-6}$ so that with (7.11) we obtain (7.8). Assume then that $n \geq 11$ and that (M, g) is not locally conformally flat. Then (2.7) shows that $\|h - c_n S_g\|_{L^\infty(2r_k)}^2 = \delta_k^4$ so that gathering (7.11) and (7.12) and since $n \geq 7$, we obtain (7.9).

Let now $1 \leq i \leq n$ be fixed. On one side, using (6.2) and since $n \geq 7$ and $|Z_{i,k,t,p}| \lesssim W_{k,t,p}$ we have that

$$\int_{M \setminus B_{y_k}(\sqrt{\delta_k})} (u + W_{k,t,p})^{2^*-3} |\phi_k(t,p)|^2 |Z_{i,k,t,p}| dv_g = o(\delta_k^{\frac{n}{2}}).$$

On the other side, using again Proposition 6.2, (2.23) and (2.7) shows that

$$\int_{B_{y_k}(\sqrt{\delta_k})} W_{k,t,p}^{2^*-3} |\phi_k(t,p)|^2 |Z_{i,k,t,p}| dv_g = o(\delta_k^{\frac{n}{2}}) + o(\delta_k^5 \mathbb{1}_{nlcf}).$$

Combining the latter estimates we get (7.10). \square

Claim 7.3. *There holds:*

$$I_{3,0} = -\frac{1}{2}(n-2)^2 (n(n-2))^{\frac{n-2}{2}} \omega_{n-1} f(\xi_0)^{1-\frac{n}{2}} u(\xi_0) \delta_k^{\frac{n-2}{2}} + o(\delta_k^{\frac{n-2}{2}}) \quad (7.13)$$

and, for any $1 \leq i \leq n$,

$$I_{3,i} = -C(n) f(\xi_0)^{-\frac{n}{2}} \nabla_i u(\xi_0) \delta_k^{\frac{n}{2}} + o(\delta_k^{\frac{n}{2}}), \quad (7.14)$$

where $C(n)$ is some explicit, positive, numerical constant only depending on n .

Proof. Write that:

$$\begin{aligned} \left| \int_{M \setminus B_{y_k}(\sqrt{\delta_k})} f \left[(u + W_{k,t,p})^{2^*-1} - u^{2^*-1} - W_{k,t,p}^{2^*-1} \right] Z_{i,k,t,p} dv_g \right| &\lesssim \int_{M \setminus B_{y_k}(\sqrt{\delta_k})} W_{k,t,p} |Z_{i,k,t,p}| dv_g \\ &\lesssim \begin{cases} o(\delta_k^{\frac{n-2}{2}}) & \text{if } i = 0, \\ o(\delta_k^{\frac{n}{2}}) & \text{if } 1 \leq i \leq n. \end{cases} \end{aligned}$$

The integral over $B_{y_k}(\sqrt{\delta_k})$ is then easily computed using the explicit expressions of $W_{k,t,p}$ and $Z_{i,k,t,p}$ given in (2.17) and (2.21). Straightforward computations give (7.13) and (7.14). \square

Claim 7.4. *There holds:*

$$I_{4,0} = o(\delta_k^{\frac{n-2}{2}}) + o(\delta_k^4 \mathbb{1}_{nlcf}), \quad (7.15)$$

and, for any $1 \leq i \leq n$:

$$I_{4,i} = o(\delta_k^{\frac{n}{2}}) + o(\delta_k^5 \mathbb{1}_{nlcf}). \quad (7.16)$$

Proof. Integrating by parts, $I_{4,i}$ is best written as:

$$I_{4,i} = \int_M \left[(\Delta_g + h) Z_{i,k,t,p} - (2^* - 1) f(y_k) W_{k,t,p}^{2^*-2} Z_{i,k,t,p} \right] \phi_k(t,p) dv_g. \quad (7.17)$$

Again, we start with $I_{4,0}$. Using (6.2), there holds:

$$\int_M \left(\delta_k^{\frac{n-2}{2}} r_k^{-n} \mathbb{1}_{r_k \leq d_{g_{y_k}} \leq 2r_k} + \delta_k^{\frac{n-2}{2}} \mathbb{1}_{nlcf} \right) |\phi_k(t,\xi)| dv_g = o(\delta_k^{\frac{n-2}{2}}).$$

Using (2.7) and (6.2) it is easily seen that there also holds:

$$\int_M (h - c_n S_g) Z_{0,k,t,p} \phi_k(t,p) dv_g = o(\delta_k^{\frac{n-2}{2}}) + o(\delta_k^4 \mathbb{1}_{nlcf}).$$

Also, (2.4) shows that $|S_{g_{y_k}}| = O(d_{g_{y_k}}(y_k, \cdot)^2)$, so that with (6.2) we obtain:

$$\int_M \Lambda_{y_k}^{2^*-2} c_n S_{g_{y_k}} Z_{0,k,t,p} \phi_k(t,p) = o(\delta_k^{\frac{n-2}{2}}) + o(\delta_k^4 \mathbb{1}_{nlcf}).$$

Combining (9.2) with (7.17) yields in the end (7.15).

Let now $1 \leq i \leq n$. Because of (2.15) and (6.2) we have:

$$\int_M \delta_k^{\frac{n}{2}} r_k^{-n-1} \mathbf{1}_{r_k \leq d_{g_{y_k}} \leq 2r_k} |\phi_k(t, p)| dv_g = o(\delta_k^{\frac{n}{2}}). \quad (7.18)$$

There holds that

$$\begin{aligned} & \int_M \left| (h - c_n S_g) Z_{i,k,t,p} \phi_k(t, p) \right| dv_g \\ &= \int_{M \setminus B_{y_k}(\sqrt{\delta_k})} |(h - c_n S_g) Z_{i,k,t,p} \phi_k(t, p)| dv_g + \int_{B_{y_k}(\sqrt{\delta_k})} |(h - c_n S_g) Z_{i,k,t,p} \phi_k(t, p)| dv_g \\ &= o(\delta_k^{\frac{n}{2}}) + o(\delta_k^5 \mathbf{1}_{n l c f}), \end{aligned} \quad (7.19)$$

where the first integral is estimated using (6.2) and the second one is estimated using Proposition 6.2 and (2.7). Assume that (M, g) is not locally conformally flat. Write again that:

$$\begin{aligned} & \int_M c_n \Lambda_{y_k} S_{g_{y_k}} Z_{i,k,t,p} \phi_k(t, p) dv_g \\ &= \int_{M \setminus B_{y_k}(\sqrt{\delta_k})} c_n \Lambda_{y_k} S_{g_{y_k}} Z_{i,k,t,p} \phi_k(t, p) dv_g + \int_{B_{y_k}(\sqrt{\delta_k})} c_n \Lambda_{y_k} S_{g_{y_k}} Z_{i,k,t,p} \phi_k(t, p) dv_g \\ &= o(\delta_k^{\frac{n}{2}}) + o(\delta_k^5 \mathbf{1}_{n l c f}), \end{aligned} \quad (7.20)$$

where as before the first integral is estimated using (6.2) and the second one using Proposition 6.2, (2.7) and (2.23). Note, as a simple computation shows, that (6.2) alone would not be enough to estimate the integral over $B_{y_k}(\sqrt{\delta_k})$ with the desired precision. Similarly, one obtains that

$$\int_M \delta_k^{\frac{n}{2}} \theta_k(y)^{2-n} |\phi_k(t, p)|(y) dv_g(y) = o(\delta_k^{\frac{n}{2}}) + o(\delta_k^5 \mathbf{1}_{n l c f}). \quad (7.21)$$

Combining (9.2) with (7.18), (7.19), (7.20) and (7.21) one obtains (7.16). \square

Let us now estimate $I_{5,i}$. For $n \geq 6$, there always holds:

$$\left| (u + W_{k,t,p})^{2^*-2} - W_{k,t,p}^{2^*-2} \right| = O(1).$$

Hence, for any $0 \leq i \leq n$:

$$|I_{5,i}| \lesssim \int_M |Z_{i,k,t,p}| |\phi_k(t, p)| dv_g.$$

Mimicking the computations that led to (7.19) one obtains, here also, that

$$I_{5,0} = o(\delta_k^{\frac{n-2}{2}}) + o(\delta_k^4 \mathbf{1}_{n l c f}) \quad (7.22)$$

and that:

$$I_{5,i} = o(\delta_k^{\frac{n}{2}}) + o(\delta_k^5 \mathbf{1}_{n l c f}). \quad (7.23)$$

By contrast with the previous integrals, $I_{6,i}$ is easily estimated using (2.23) and (6.2). Straightforward computations give indeed that for any $0 \leq i \leq n$:

$$I_{6,i} = o(\delta_k^{\frac{n}{2}}). \quad (7.24)$$

Claim 7.5. *There holds that:*

$$I_{7,i} = \begin{cases} o(\delta_k^{\frac{n-2}{2}}) & \text{if } i = 0 \\ o(\delta_k^{\frac{n}{2}}) & \text{if } 1 \leq i \leq n. \end{cases} \quad (7.25)$$

Proof. For any $0 \leq i \leq n$, we write:

$$\begin{aligned} I_{7,i} &= \int_M (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(u^{-2^*-1} - (u + W_{k,t,p} + \phi_k(t,p))^{-2^*-1} \right) Z_{i,k,t,p} dv_g \\ &\quad + \int_M (u + W_{k,t,p} + \phi_k(t,p))^{-2^*-1} (|\mathcal{L}_g T + \sigma|_g^2 - |\mathcal{L}_g T_{k,t,p} + \sigma|_g^2) Z_{i,k,t,p} dv_g \\ &:= I_{7,i}^1 + I_{7,i}^2. \end{aligned} \quad (7.26)$$

We split the integral $I_{7,i}^1$ into an integral in $B_{y_k}(\sqrt{\delta_k})$ and another in $M \setminus B_{y_k}(\sqrt{\delta_k})$. Since the integrand of $I_{7,i}^1$ is bounded by $|Z_{i,k,t,p}|$ in $B_{y_k}(\sqrt{\delta_k})$ (up to some positive constant that does not depend on k) and since

$$\left| u^{-2^*-1} - (u + W_{k,t,p} + \phi_k(t,p))^{-2^*-1} \right| \lesssim W_{k,t,p} + |\phi_k(t,p)| \text{ in } M \setminus B_{y_k}(\sqrt{\delta_k}),$$

there holds with (6.2) that:

$$I_{7,0}^1 = o(\delta_k^{\frac{n-2}{2}}), \quad (7.27)$$

and that, for $1 \leq i \leq n$,

$$I_{7,i}^1 = o(\delta_k^{\frac{n}{2}}). \quad (7.28)$$

To compute $I_{7,i}^2$, we again split the integration domain into $B_{y_k}(\sqrt{\delta_k})$ and $M \setminus B_{y_k}(\sqrt{\delta_k})$. On the one side, using (6.2) and (9.11), we obtain that:

$$\begin{aligned} &\int_{B_{y_k}(\sqrt{\delta_k})} (u + W_{k,t,p} + \phi_k(t,p))^{-2^*-1} (|\mathcal{L}_g T + \sigma|_g^2 - |\mathcal{L}_g T_{k,t,p} + \sigma|_g^2) Z_{i,k,t,p} dv_g \\ &= \begin{cases} o(\delta_k^{\frac{n-2}{2}}) & \text{if } i = 0 \\ o(\delta_k^{\frac{n}{2}}) & \text{if } 1 \leq i \leq n. \end{cases} \end{aligned} \quad (7.29)$$

On the other side, using again (9.11), we get:

$$\begin{aligned} &\int_{M \setminus B_{y_k}(\sqrt{\delta_k})} (u + W_{k,t,p} + \phi_k(t,p))^{-2^*-1} (|\mathcal{L}_g T + \sigma|_g^2 - |\mathcal{L}_g T_{k,t,p} + \sigma|_g^2) Z_{i,k,t,p} dv_g \\ &= \begin{cases} o(\delta_k^{\frac{n-2}{2}}) & \text{if } i = 0 \\ o(\delta_k^{\frac{n}{2}}) & \text{if } 1 \leq i \leq n. \end{cases} \end{aligned} \quad (7.30)$$

Combining (7.26)–(7.30) gives (7.25). \square

7.2. The $n = 6$ case. In the 6-dimensional case we need to take into account the compensation phenomenon for system (1.1), since we need to push the asymptotic estimates of the $\lambda_k^i(t,p)$ one order

further. For this, we write the scalar equation of (7.2) as:

$$\begin{aligned} \sum_{i=0}^6 \lambda_k^i(t, p) (\triangle_g + h) Z_{i,k,t,p} = & \\ & + (\triangle_g + h - 2fu) W_{k,t,p} - f(y_k) W_{k,t,p}^2 + (f(y_k) - f) W_{k,t,p}^2 \\ & + f\phi_k(t, p)^2 \\ & + (\triangle_g + h - 2fu) \phi_k(t, p) - 2f(y_k) W_{k,t,p} \phi_k(t, p) \\ & + 2(f(y_k) - f) W_{k,t,p} \phi_k(t, p) \\ & + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^4} - \frac{|\mathcal{L}_g T_{k,t,p} + \sigma|_g^2 + \pi^2}{(u + W_{k,t,p} + \phi_k(t, p))^4}, \end{aligned}$$

and we let, for $0 \leq i \leq n$:

$$\begin{aligned} J_{1,i} &= \int_M \left[(\triangle_g + h - 2fu) W_{k,t,p} - f(y_k) W_{k,t,p}^2 + (f(y_k) - f) W_{k,t,p}^2 \right] Z_{i,k,t,p} dv_g, \\ J_{2,i} &= \int_M f\phi_k(t, p)^2 Z_{i,k,t,p} dv_g, \\ J_{3,i} &= \int_M \left[(\triangle_g + h - 2fu) \phi_k(t, p) - 2f(y_k) W_{k,t,p} \phi_k(t, p) \right] Z_{i,k,t,p} dv_g, \\ J_{4,i} &= \int_M 2(f(y_k) - f) W_{k,t,p} \phi_k(t, p) Z_{i,k,t,p} dv_g, \\ J_{5,i} &= \int_M \left(\frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^4} - \frac{|\mathcal{L}_g T_{k,t,p} + \sigma|_g^2 + \pi^2}{(u + W_{k,t,p} + \phi_k(t, p))^4} \right) Z_{i,k,t,p} dv_g. \end{aligned}$$

These integrals are again computed in a series of Claims. The situation in the 6-dimensional case is different, since an additional contribution in the expansion of the $\lambda_k^i(t, p)$ comes from the integral $J_{5,i}$.

The computations of $J_{1,i}$ and $J_{2,i}$ follows the exact same lines than in the $n \geq 7$ case. Using (2.17), (2.21), (2.23), (2.15) and (6.2) it is esily seen that there holds:

$$J_{1,0} = -5K_6^{-6} f(\xi_0)^{-3} H(p) \tau_k \delta_k^2 + o(\delta_k^3), \quad (7.31)$$

and, for $1 \leq i \leq 6$,

$$J_{1,i} = -30K_6^{-6} f(\xi_0)^{-3} \nabla_i H(p) \frac{\tau_k}{\beta_k} \delta_k^3 + o(\delta_k^4), \quad (7.32)$$

where K_6^{-6} is defined in (7.7). Also, mimicking the computations that led to (7.10) gives here as well:

$$J_{2,i} = \begin{cases} o(\delta_k^3), & \text{if } i = 0, \\ o(\delta_k^4), & \text{if } 1 \leq i \leq 6. \end{cases} \quad (7.33)$$

Claim 7.6. *There holds:*

$$J_{3,0} = o(\delta_k^3), \quad (7.34)$$

and, for any $1 \leq i \leq 6$:

$$J_{3,i} = o(\delta_k^4). \quad (7.35)$$

Proof. As for the $n \geq 7$ case, we rewrite $J_{3,i}$ as:

$$J_{3,i} = \int_M \left([\Delta_g + (h - 2fu)] Z_{i,k,t,p} - 2f(y_k) W_{k,t,p} Z_{i,k,t,p} \right) \phi_k(t, p) dv_g.$$

Compared to the $n \geq 7$ case, the estimation of $J_{3,i}$ requires the additional information on $\phi_k(t, p)$ given by Proposition 6.3. Namely, using (6.24) for a suitable choice of a sequence $(R_k)_k$ yields:

$$\int_M \left(\delta_k^2 r_k^{-6} \mathbb{1}_{r_k \leq d_{g_{y_k}} \leq 2r_k} + \delta_k^2 \mathbb{1}_{n_{lcf}} \right) \phi_k(t, p) dv_g = o(\delta_k^3),$$

while using (2.7) and (6.2) gives:

$$\int_M \left(h - \frac{1}{5} S_g - 2fu \right) Z_{0,k,t,p} \phi_k(t, p) dv_g = o(\delta_k^3).$$

If (M, g) is not locally conformally flat there holds, for some R_k satisfying $R_k \sqrt{\delta_k} = o(1)$:

$$\begin{aligned} & \int_M \frac{1}{5} \Lambda_{y_k} S_{g_{y_k}} Z_{0,k,t,p} \phi_k(t, p) dv_g \\ &= \int_{M \setminus B_{y_k}(R_k \sqrt{\delta_k})} \frac{1}{5} \Lambda_{y_k} S_{g_{y_k}} Z_{0,k,t,p} \phi_k(t, p) dv_g + \int_{B_{y_k}(R_k \sqrt{\delta_k}) \setminus B_{y_k}(\sqrt{\delta_k})} \frac{1}{5} \Lambda_{y_k} S_{g_{y_k}} Z_{0,k,t,p} \phi_k(t, p) dv_g \\ &+ \int_{B_{y_k}(\sqrt{\delta_k})} \frac{1}{5} \Lambda_{y_k} S_{g_{y_k}} Z_{0,k,t,p} \phi_k(t, p) dv_g \\ &= o(\delta_k^3), \end{aligned} \tag{7.36}$$

where we used again (2.4) and we estimated the first integral using (6.24), the second one using (6.2) and the third one using Proposition 6.2. With (9.2) we therefore obtain (7.34).

Let now $1 \leq i \leq 6$. Similarly, using (6.24) for a suitable radius R_k and using (2.15) yields:

$$\int_M \delta_k^3 r_k^{-7} |\phi_k(t, \xi)| dv_g = o(\delta_k^4),$$

and as in (7.36) (6.24), (6.2), Proposition 6.2 and (2.4) show that if (M, g) is not locally conformally flat, then:

$$\int_M \frac{1}{5} \Lambda_{y_k} S_{g_{y_k}} Z_{i,k,t,p} \phi_k(t, p) dv_g = o(\delta_k^4).$$

Independently, there holds that:

$$\begin{aligned} & \int_M \left(h - \frac{1}{5} S_g - 2fu \right) Z_{i,k,t,p} \phi_k(t, p) dv_g \\ &= \int_{B_{y_k}(\sqrt{\delta_k})} \left(h - \frac{1}{5} S_g - 2fu \right) Z_{i,k,t,p} \phi_k(t, p) dv_g \\ &+ \int_{M \setminus B_{y_k}(\sqrt{\delta_k})} \left(h - \frac{1}{5} S_g - 2fu \right) Z_{i,k,t,p} \phi_k(t, p) dv_g \\ &= o(\delta_k^4), \end{aligned}$$

where we used (2.7) and Proposition 6.2 to estimate the first integral and (2.7) and (6.2) to estimate the second one. Finally, in case (M, g) is not locally conformally flat, mimicking the computations that led to (7.36) we get that:

$$\int_M \delta_k^3 \theta_k(\cdot)^{-4} |\phi_k(t, p)| dv_g = o(\delta_k^4).$$

Combining the above estimates with (9.2) gives in the end (7.35). \square

Here again, $J_{4,i}$ is easily estimated : straightforward computations using (2.23) give indeed that, for any $0 \leq i \leq 6$:

$$J_{4,i} = o(\delta_k^4). \quad (7.37)$$

As already mentioned, and unlike in the higher-dimensional case, the coupling field X enters the expansion of $J_{5,i}$ for $n = 6$:

Claim 7.7. *There holds:*

$$J_{5,0} = \kappa \delta_k^3 + o(\delta_k^3), \quad (7.38)$$

where the constant κ is explicitly given by (7.45) below, and, for any $1 \leq i \leq 6$:

$$J_{5,i} = O(\delta_k^{\frac{7}{2}}). \quad (7.39)$$

Proof. We again write that there holds, for any $0 \leq i \leq 6$:

$$\begin{aligned} J_{5,i} &= \int_M (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(u^{-4} - (u + W_{k,t,p} + \phi_k(t,p))^{-4} \right) Z_{i,k,t,p} dv_g \\ &\quad + \int_M (u + W_{k,t,p} + \phi_k(t,p))^{-4} (|\mathcal{L}_g T + \sigma|_g^2 - |\mathcal{L}_g T_{k,t,p} + \sigma|_g^2) Z_{i,k,t,p} dv_g \\ &:= J_{5,i}^1 + J_{5,i}^2. \end{aligned} \quad (7.40)$$

Let $R > 0$. Since there holds

$$\left| u^{-4} - (u + W_{k,t,p} + \phi_k(t,p))^{-4} \right| \lesssim (W_{k,t,p} + |\phi_k(t,p)|) \text{ in } M \setminus B_{y_k}(R\sqrt{\delta_k}),$$

then (6.24) shows that:

$$\left| \int_{M \setminus B_{y_k}(R\sqrt{\delta_k})} (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(u^{-4} - (u + W_{k,t,p} + \phi_k(t,p))^{-4} \right) Z_{0,k,t,p} dv_g \right| \lesssim \frac{\delta_k^3}{R^2} + o(\delta_k^3). \quad (7.41)$$

Independently, by (6.2), Lebesgue's dominated convergence theorem shows that there holds:

$$\begin{aligned} &\delta_k^{-3} \int_{B_{y_k}(R\sqrt{\delta_k})} (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) \left(u^{-4} - (u + W_{k,t,p} + \phi_k(t,p))^{-4} \right) Z_{0,k,t,p} dv_g \\ &= \frac{(24)^2}{f(\xi_0)} (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2)(\xi_0) \int_{B_0(R)} \left[u(\xi_0)^{-4} - \left(u(\xi_0) + \left(\frac{24}{f(\xi_0)} \right)^2 |y|^{-4} \right)^4 \right] |y|^{-4} dy + o(1), \end{aligned} \quad (7.42)$$

as $k \rightarrow +\infty$, so that (7.41) and (7.42) together show that there holds

$$\begin{aligned} J_{5,0}^1 &= \frac{(24)^2}{f(\xi_0)} (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2)(\xi_0) \\ &\quad \times \int_{\mathbb{R}^6} \left[u(\xi_0)^{-4} - \left(u(\xi_0) + \left(\frac{24}{f(\xi_0)} \right)^2 |y|^{-4} \right)^4 \right] |y|^{-4} dy \cdot \delta_k^3 + o(\delta_k^3). \end{aligned} \quad (7.43)$$

Using (9.10) with (6.3) one gets that:

$$|\mathcal{L}_g T + \sigma|_g^2 - |\mathcal{L}_g T_{k,t,\xi} + \sigma|_g^2 = -|\mathcal{L}_g \Theta_k|_g^2 + O(|\mathcal{L}_g \Theta_k|_g) + O(\delta_k),$$

where Θ_k is defined in (9.9) below. Using the asymptotic (9.17) together with (2.24) and the dominated convergence theorem show that:

$$\begin{aligned} J_{5,0}^2 &= -C(6)f(\xi_0)^{-8}\alpha^2 \\ &\quad \times \int_{\mathbb{R}^6} \left(u(\xi_0) + \left(\frac{f(\xi_0)}{24} \right)^{-2} |y|^{-4} \right)^{-4} \left(2 + 28 \left| \left\langle \frac{Z(0)}{|Z(0)|_{eucl}}, \frac{y}{|y|} \right\rangle \right|^2 \right) |y|^{-14} dy \cdot \delta_k^3, \end{aligned} \quad (7.44)$$

where we have let $C(6) = 2^{32}3^{95}5^{-4}$ and where α is as in (2.25). Combining (7.43) and (7.44) in (7.40) one obtains in the end (7.38), where κ is given by:

$$\begin{aligned} \kappa = & \frac{(24)^2}{f(\xi_0)} (|\mathcal{L}_g T + \sigma|_g^2 + \pi^2)(\xi_0) \times \int_{\mathbb{R}^6} \left[u(\xi_0)^{-4} - \left(u(\xi_0) + \left(\frac{24}{f(\xi_0)} \right)^2 |y|^{-4} \right)^4 \right] |y|^{-4} dy \\ & - C(6) f(\xi_0)^{-8} \alpha^2 \times \int_{\mathbb{R}^6} \left(u(\xi_0) + \left(\frac{f(\xi_0)}{24} \right)^{-2} |y|^{-4} \right)^{-4} \left(2 + 28 \left| \left\langle \frac{Z(0)}{|Z(0)|_{eucl}}, \frac{y}{|y|} \right\rangle \right|^2 \right) |y|^{-14} dy. \end{aligned} \quad (7.45)$$

In particular, up to choosing α as in (2.25) small enough, we have $\kappa > 0$.

Let now $1 \leq i \leq 6$. Mimicking the proof of (7.41) and (7.42) gives, by (6.24) and the dominated convergence theorem, that

$$J_{5,i}^1 = o(\delta_k^{\frac{7}{2}}).$$

Finally, using the expansion (9.10) along with (9.17) shows, again by dominated convergence, that there holds:

$$J_{5,i}^2 = O(\delta_k^{\frac{7}{2}}).$$

With (7.40) we obtain that (7.39) holds. \square

8. CONCLUSIVE ARGUMENT

In this section we conclude the proof of Theorem 1.1. We use the expansions of the $\lambda_k^i(t, p)$ obtained in Section 7 to show that, for any k , there exist $(t_k, p_k) \in [1/D, D] \times \overline{B_0(1)}$ such that $\lambda_k^i(t_k, p_k) = 0$ for any $0 \leq i \leq n$. We use here the notations of Section 7. In particular, δ_k and y_k are defined by (2.16) and (7.1).

First, for any $0 \leq i \leq n$, there holds:

$$\int_M \langle \nabla Z_{i,k,t,p}, \nabla Z_{j,k,t,p} \rangle_g + h Z_{i,k,t,p} Z_{j,k,t,p} dv_g = \delta_{ij} \|\nabla V_{i,y_k}\|_{L^2(\mathbb{R}^n)}^2 + O(\delta_k), \quad (8.1)$$

uniformly in the choice of h, t and p , and where V_{i,y_k} is defined in (2.18).

Assume first that (M, g) is locally conformally flat or that $7 \leq n \leq 10$. Then, combining (7.5), (7.6), (7.8), (7.9), (7.10), (7.13), (7.14), (7.15), (7.16), (7.22), (7.23), (7.24) and (7.25) in (7.4) yields, with (8.1) and (2.6), that:

$$\begin{aligned} & (I_{n+1} + O(\delta_k)) \begin{pmatrix} \lambda_k^0(t, p) \\ \vdots \\ \lambda_k^n(t, p) \end{pmatrix} \\ &= \begin{pmatrix} \mu_k^{\frac{n-2}{2}} \left[\frac{8(n-1)}{(n-2)(n-4)} K_n^{-n} f(\xi_0)^{-\frac{n}{2}} H(p) t^2 - \frac{1}{2} (n-2)^2 (n(n-2))^{\frac{n-2}{2}} \omega_{n-1} f(\xi_0)^{1-\frac{n}{2}} u(\xi_0) t^{\frac{n-2}{2}} \right. \\ \qquad \qquad \qquad \left. - \frac{10}{9} f(\xi_0)^{-4} K_{10}^{-10} |W_g(\xi)|_g^2 t^4 \mathbf{1}_{n=10} + R_k^0(t, p) \right] \\ \mu_k^{\frac{n}{2}} \left[\frac{2n(n-1)}{\beta_k} K_n^{-n} f(\xi_0)^{-\frac{n}{2}} \nabla_i H(p) t^3 + R_k^i(t, p) \right] \end{pmatrix}, \end{aligned} \quad (8.2)$$

where, for $0 \leq i \leq n$, $R_k^i(t, p)$ denotes a function which converges to zero in $C^0([1/D, D] \times \overline{B_0(1)})$ as $k \rightarrow +\infty$, where K_n^{-n} is as in (7.7) and where β_k is as in (2.6). The continuity of R_k^i , $0 \leq i \leq n$,

is a direct consequence of the continuity of ϕ_k as stated in Proposition 6.1. Let F be the function defined in $[1/D, D] \times \overline{B_0(1)}$ by:

$$F(t, p) = \begin{pmatrix} \frac{8(n-1)}{(n-2)(n-4)} K_n^{-n} f(\xi_0)^{-\frac{n}{2}} H(p) t^2 - \frac{1}{2} (n-2)^2 (n(n-2))^{\frac{n-2}{2}} \omega_{n-1} f(\xi_0)^{1-\frac{n}{2}} u(\xi_0) t^{\frac{n-2}{2}} \\ - \frac{10}{9} f(\xi_0)^{-4} K_{10}^{-10} |W_g(\xi)|_g^2 t^4 \mathbb{1}_{n=10} \\ \frac{2n(n-1)}{n-4} K_n^{-n} f(\xi_0)^{-\frac{n}{2}} \nabla_i H(p) t^3 \end{pmatrix}$$

and let $t_0 > 0$ be the unique solution of:

$$\begin{aligned} \frac{8(n-1)}{(n-2)(n-4)} K_n^{-n} f(\xi_0)^{-\frac{n}{2}} t^2 &= \frac{1}{2} (n-2)^2 (n(n-2))^{\frac{n-2}{2}} \omega_{n-1} f(\xi_0)^{1-\frac{n}{2}} u(\xi_0) t^{\frac{n-2}{2}} \\ &+ \frac{10}{9} f(\xi_0)^{-4} K_{10}^{-10} |W_g(\xi)|_g^2 t^4 \mathbb{1}_{n=10}. \end{aligned}$$

Using the assumption that 0 is a non-degenerate critical point of H , it is easily checked that the differential of F at $(t_0, 0)$ is invertible. Since there holds $F(t_0, 0) = 0$ by definition of t_0 , and since the $R_k^i(t, p)$ appearing in (8.2) uniformly converge to 0 as $k \rightarrow +\infty$, standard degree-theoretic arguments yield the existence of a sequence $(t_k, p_k) \in]1/D, D[\times B_0(1)$ of interior points such that

$$F(t_k, p_k) + \begin{pmatrix} R_k^0(t_k, p_k) \\ R_k^i(t_k, p_k)_{1 \leq i \leq n} \end{pmatrix} = 0$$

for any k . Note that throughout this argument we assumed that D is chosen large enough to have $t_0 \in [2/D, D/2]$. Coming back to (8.2), this amounts to say that $\lambda_k^i(t_k, p_k) = 0$ for any $0 \leq i \leq n$. And with Proposition 6.1 and (7.2) this shows that the function u_{k, t_k, p_k} is a solution of system (1.1), and concludes the proof of Theorem 1.1 in this case.

If now we assume that $n \geq 11$ and (M, g) is not locally conformally flat, the same arguments lead to the following expansion for the $\lambda_k^i(t, p)$:

$$\begin{aligned} & \left(I_{n+1} + O(\delta_k) \right) \begin{pmatrix} \lambda_k^0(t, p) \\ \vdots \\ \lambda_k^n(t, p) \end{pmatrix} \\ &= K_n^{-n} f(\xi_0)^{-\frac{n}{2}} \begin{pmatrix} \mu_k^4 \left[\frac{8(n-1)}{(n-2)(n-4)} H(p) t^2 - \frac{1}{3} \frac{n(n-2)}{(n-4)(n-6)} f(\xi_0) |W_g(\xi_0)|_g^2 t^4 + R_k^0(t, p) \right] \\ \frac{\mu_k^5}{\beta_k} \left[\frac{2n(n-1)}{n-4} \nabla_i H(p) t^3 + R_k^i(t, p) \right] \end{pmatrix}. \end{aligned}$$

While in the 6-dimensional case, we end up with:

$$\left(I_7 + O(\delta_k) \right) \begin{pmatrix} \lambda_k^0(t, p) \\ \vdots \\ \lambda_k^6(t, p) \end{pmatrix} = \begin{pmatrix} \mu_k^3 \left[-5K_6^{-6} f(\xi_0)^{-3} H(p) t^2 + \kappa t^3 + R_k^0(t, p) \right] \\ \frac{\mu_k^4}{\beta_k} \left[-30K_6^{-6} f(\xi_0)^{-3} \nabla_i H(p) t^3 + R_k^i(t, p) \right] \end{pmatrix},$$

where the constant κ is defined in (7.45). The conclusion in these cases follows then from the exact same arguments than in the previous case, thus concluding the proof of Theorem 1.1.

9. TECHNICAL RESULTS

9.1. Conformal laplacian of $W_{k,t,\xi}$ and of the $Z_{i,k,t,\xi}$, $0 \leq i \leq n$. Below are given the expressions of the conformal laplacian of the functions W_k and $Z_{i,k,t,\xi}$ defined in (2.17) and (2.21). Let $D > 0$ and let $(t_k, \xi_k)_k$ be a sequence of points in $[1/D, D] \times M$. Then, for any function $h \in C^\infty(M)$, there holds:

$$\begin{aligned} (\Delta_g + h) W_{k,t_k,\xi_k} &= f(\xi_k) W_{k,t_k,\xi_k}^{2^*-1} + (h - c_n S_g) W_{k,t_k,\xi_k} + c_n \Lambda_{\xi_k}^{2^*-2} S_{g_{\xi_k}} W_{k,t_k,\xi_k} \\ &\quad + O(\delta_k^{\frac{n-2}{2}} \mathbb{1}_{d_k \leq 2r_k}) + O(\delta_k^{\frac{n-2}{2}} r_k^{-n} \mathbb{1}_{r_k \leq d_k \leq 2r_k}), \end{aligned} \quad (9.1)$$

and, for any $1 \leq i \leq n$:

$$\begin{aligned} (\Delta_g + h) Z_{0,k,t_k,\xi_k} &= (2^* - 1) f(\xi_k) W_{k,t_k,\xi_k}^{2^*-2} Z_{0,k,t_k,\xi_k} + (h - c_n S_g) Z_{0,k,t_k,\xi_k} \\ &\quad + c_n \Lambda_{\xi_k}^{2^*-2} S_{g_{\xi_k}} Z_{0,k,t_k,\xi_k} + O(\delta_k^{\frac{n-2}{2}} r_k^{-n} \mathbb{1}_{r_k \leq d_{g_{\xi_k}} \leq 2r_k}) + O(\delta_k^{\frac{n-2}{2}} \mathbb{1}_{n_{lcf}, d_{g_{\xi_k}} \leq 2r_k}), \\ (\Delta_g + h) Z_{i,k,t_k,\xi_k} &= (2^* - 1) f(\xi_k) W_{k,t_k,\xi_k}^{2^*-2} Z_{i,k,t_k,\xi_k} + (h - c_n S_g) Z_{i,k,t_k,\xi_k} \\ &\quad + c_n \Lambda_{\xi_k}^{2^*-2} S_{g_{\xi_k}} Z_{i,k,t_k,\xi_k} + O(\delta_k^{\frac{n}{2}} r_k^{-n-1} \mathbb{1}_{r_k \leq d_{g_{\xi_k}} \leq 2r_k}) + O(\delta_k^{\frac{n}{2}} \theta_k(\cdot)^{2-n} \mathbb{1}_{n_{lcf}, d_{g_{\xi_k}} \leq 2r_k}). \end{aligned} \quad (9.2)$$

In (9.1) and (9.2) Λ_{ξ_k} is as in (2.3) and $d_k = d_{g_{\xi_k}}(\xi_k, \cdot)$. The notation “ $O(f)$ ” denotes a smooth function which can be uniformly bounded in $C^0(M)$ by $C_0|f|$, where C_0 is some positive constant that does not depend on k . Also, the notational shorthand $\mathbb{1}_{n_{lcf}}$ is used to indicate that the corresponding term vanishes if (M, g) is locally conformally flat in a fixed neighbourhood of ξ_k . The notations $\mathbb{1}_{r_k \leq d_{g_{\xi_k}} \leq 2r_k}$ and $\mathbb{1}_{n_{lcf}, d_{g_{\xi_k}} \leq 2r_k}$ are defined similarly.

These expressions are obtained from (2.17) and (2.21), using the conformal invariance property of the conformal laplacian and the properties of the normal conformal factor Λ_{ξ_k} . See for instance Esposito-Pistoia-Vétois [23] and Lee-Parker [32] for more details.

9.2. Pointwise estimates for solutions of the 1-form equation. Let (M, g) be a closed Riemannian manifold possessing no conformal Killing fields. This means that the operator $\vec{\Delta}_g$, when acting on H^1 fields of 1-forms, has zero kernel. We start by recalling that the operator $\vec{\Delta}_g$ always possesses local Green fields:

Proposition 9.1. *Let $x_0 \in M$ and $\delta < \frac{i_g(M)}{2}$. For any $x \in B_{x_0}(\delta)$ there exist n fields of 1-forms $G_1(x, \cdot), \dots, G_n(x, \cdot)$ defined in $B_{x_0}(\delta) \setminus \{x\}$ which form a fundamental solution for the operator $\vec{\Delta}_g$ in $B_{x_0}(\delta)$ in the following sense: for any 1-form $X \in \Gamma(TB_{x_0}(\delta))$ with $X \equiv 0$ on $\partial B_{x_0}(\delta)$, there holds:*

$$(X - \pi(X))_i(x) = \int_{B_{x_0}(\delta)} \left\langle G_i(x, y), \vec{\Delta}_g X(y) \right\rangle_{g(y)} dv_g(y), \quad (9.3)$$

where π denotes the orthogonal projection for the $L^2(B_{x_0}(\delta))$ -scalar product on the set

$$\{X \in H^1(B_{x_0}(\delta)), \mathcal{L}_g X = 0 \text{ in } B_{x_0}(\delta)\},$$

and where the coordinates in (9.3) are taken in any chart in $B_{x_0}(\delta)$. The Green fields satisfy in addition: for any $1 \leq i \leq n$, for any $x \in B_{x_0}(\delta/2)$, $y \in B_{x_0}(\delta)$,

$$d_g(x, y) |\nabla G_i(x, y)|_g + |G_i(x, y)|_g \leq C(n, g) d_g(x, y)^{2-n}, \quad (9.4)$$

where the derivative in (9.4) can be taken with respect to x or to y .

The construction of these Green fields is carried out in the Euclidean case in Druet-Premoselli [21] in dimension 3 and in Premoselli [39]. The extension to the Riemannian case shows no additional

difficulty. The proof just consists in adapting the steps that lead to the construction of the Riemannian Green function for the Laplace-Beltrami operator on a Riemannian manifold starting from the euclidean one (see for instance Appendix A in Druet-Hebey-Robert [18] or Robert [43]), and we will not detail it here. Using the Euclidean expression computed in Premoselli [39] one finds that these Green fields G_i have the following expansion:

$$G_i(x, \exp_x(y))_j = -\frac{1}{4(n-1)\omega_{n-1}}|y|^{2-n} \left((3n-2)\delta_{ij} + (n-2)\frac{y_i y_j}{|y|^2} \right) \left(1 + O(|y|) \right), \quad (9.5)$$

for any $x \in M$, where the exponential map is taken here for the background metric g and ω_{n-1} is the volume of the standard $(n-1)$ -sphere in \mathbb{R}^n . If X is a smooth 1-form in $B_{x_0}(\delta)$ which vanishes on $\partial B_{x_0}(\delta)$ formula (9.3) can be differentiated to obtain: for any $1 \leq i, j \leq n$, for any $x \in B_{x_0}(\delta)$,

$$\mathcal{L}_g X_{ij}(x) = \int_{B_{x_0}(\delta)} \left\langle H_{ij}(x, y), \vec{\Delta}_g X(y) \right\rangle_{g(y)} dv_g(y), \quad (9.6)$$

where we have let:

$$H_{ij}(x) = \nabla_i G_j(x, y) + \nabla_j G_i(x, y) - \frac{2}{n} g^{kl}(x) \nabla_k G_l(x, y) g_{ij}(x), \quad (9.7)$$

and the covariant derivatives are all taken here with respect to x .

We now use these Green fields to derive optimal pointwise estimates on solutions of the 1-form equation. Let u be a smooth positive function in M . Let $D > 0$ and let $(t_k, \xi_k)_k \in [\frac{1}{D}, D] \times M$ be a sequence of points, and consider the function W_{k, t_k, ξ_k} given by (2.17), where δ_k is given by (2.16) and $(\mu_k)_k$ denotes some sequence of positive numbers which converge to zero. Recall that the functions W_{k, t_k, ξ_k} – that shall now be abbreviated as W_k – are compactly supported on a ball centered at ξ_k and of radius $2r_k$ given by (2.15) (the ball is taken here with respect to the metric g_{ξ_k}). Let $(\varepsilon_k)_k$ denote a sequence of positive numbers and, for any k , let v_k be a continuous function satisfying

$$\left\| \frac{v}{u + W_k} \right\|_{C^0(M)} \leq \varepsilon_k. \quad (9.8)$$

Let X and Y be two 1-forms in M and, for any k , let T_k, Θ_k and T be the unique solutions of the following equations in M :

$$\begin{aligned} \vec{\Delta}_g T_k &= (u + W_k + v_k)^{2^*} X + Y, \\ \vec{\Delta}_g \Theta_k &= W_k^{2^*} X, \\ \vec{\Delta}_g T &= u^{2^*} X + Y. \end{aligned} \quad (9.9)$$

Then the following pointwise asymptotic estimate on $\mathcal{L}_g T_k$ holds:

Proposition 9.2. *For any sequence $(x_k)_k$ of points in M , there holds:*

$$\mathcal{L}_g T_k(x_k) = \left(1 + O(\delta_k) \right) \mathcal{L}_g \Theta_k(x_k) + \mathcal{L}_g T(x_k) + O\left(\|X\|_{C^0(M)} \varepsilon_k + |X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} \right), \quad (9.10)$$

uniformly in k . As a consequence, there holds, for any $x \in M$:

$$|\mathcal{L}_g T_k - \mathcal{L}_g T|_g(x) \leq C \left(\left[|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} \right] \theta_k(x)^{1-n} + \|X\|_{C^0(M)} \varepsilon_k \right), \quad (9.11)$$

where C is a positive constant independent of k and $\|X\|_{C^0(M)}$ and where $\theta_k(x)$ is as in (4.25).

Proof. Let $G_i, 1 \leq i \leq n$, be the family of Green fields given by Proposition 9.1 and defined in $B_{\xi_k}(\delta) \times B_{\xi_k}(\delta) \setminus \{x = y\}$. Let $\eta \in C_c^\infty(\mathbb{R}^+)$ be a smooth cut-off function equal to 1 in $[0, \delta/4]$ and equal to 0 outside of $[0, \delta/2]$. We define the following 1-forms, compactly supported in $B_{\xi_k}(\delta/2)$:

$$P_k(x)_i = \left(\int_{B_{x_0}(\frac{\delta}{2})} \left\langle G_i(x, y), \left[(u + W_k + v_k)^{2^*} - u^{2^*} \right] (y) X(y) \right\rangle_{g(y)} dv_g(y) \right) \cdot \eta(d_g(\xi_k, x)). \quad (9.12)$$

By definition of G_i , by the choice of η , by (9.4), (9.8) and by Giraud's lemma there holds, for any $x \in B_{\xi_k}(\delta/2)$:

$$\begin{aligned} |P_k|(x) &\leq C \left(|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} \right) \theta_k(x)^{2-n} + C \|X\|_{C^0(M)} \varepsilon_k \\ |\mathcal{L}_g P_k|_g(x) &\leq C \left(|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} \right) \theta_k(x)^{1-n} + C \|X\|_{C^0(M)} \varepsilon_k. \end{aligned} \quad (9.13)$$

In (9.13) and until the end of this section C will denote some positive constant that does not depend on k or on $\|X\|_{C^0(M)}$. Since $\left| (u + W_k + v_k)^{2^*} - u^{2^*} \right| \leq C \varepsilon_k$ in $M \setminus B_{\xi_k}(\delta/4)$, and by definition of G_i , one obtains with (9.9), (9.12) and (9.13) that:

$$\begin{aligned} \vec{\Delta}_g(T_k - P_k - T) &= 0 \text{ in } B_{\xi_k}(\delta/4) \\ \left| \vec{\Delta}_g(T_k - P_k - T) \right| &\leq C \left(|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} + \|X\|_{C^0(M)} \varepsilon_k \right) \text{ in } B_{\xi_k}(\delta/2) \setminus B_{\xi_k}(\delta/4) \\ \left| \vec{\Delta}_g(T_k - P_k - T) \right| &\leq C \|X\|_{C^0(M)} \varepsilon_k \text{ in } M \setminus B_{\xi_k}(\delta/2). \end{aligned}$$

Standard elliptic regularity theory applies for $\vec{\Delta}_g$ (see e.g. Premoselli [39]), and with the latter estimates shows that there holds, in $C^0(M)$:

$$\mathcal{L}_g T_k = \mathcal{L}_g P_k + \mathcal{L}_g T + O \left(|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} + \|X\|_{C^0(M)} \varepsilon_k \right). \quad (9.14)$$

Independently, similar arguments show that there holds:

$$\mathcal{L}_g \Theta_k = \mathcal{L}_g Q_k + O \left(|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} \right), \quad (9.15)$$

where Θ_k is defined in (9.9) and where we have let, for $1 \leq i \leq n$:

$$Q_k(x)_i = \left(\int_{B_{x_0}(\frac{\delta}{2})} \left\langle G_i(x, y), W_k^{2^*}(y) X(y) \right\rangle_{g(y)} dv_g(y) \right) \cdot \eta(d_g(\xi_k, x)). \quad (9.16)$$

Since there holds, in $B_{\xi_k}(\delta/4)$, that

$$\left| (u + W_k + v_k)^{2^*} - u^{2^*} - W_k^{2^*} \right| \leq C \left(\varepsilon_k + W_k^{2^*-1} + W_k \right),$$

we obtain with Giraud's lemma, (9.12), (9.15) and (9.16) that, for any sequence $(x_k)_k$ in M :

$$\mathcal{L}_g P_k(x_k) = \left(1 + O(\delta_k) \right) \mathcal{L}_g \Theta_k(x_k) + O \left(|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} + \|X\|_{C^0(M)} \varepsilon_k \right).$$

Then estimate (9.10) then follows from the latter expansion and (9.14), and (9.11) follows from (9.10). \square

Using (9.5), (9.6), (9.7), (9.15) and (9.16) one obtains the following asymptotic estimate for $\mathcal{L}_g \Theta_k$: for any sequence $(x_k)_k$ of points in M satisfying $\delta_k \ll d_g(x_k, \xi_k) \leq \delta/4$ there holds:

$$\begin{aligned} \mathcal{L}_\xi \Theta_k(x_k)_{ij} = & K(n) f(\xi_k)^{-\frac{n}{2}} \left[\delta_{ij} \zeta^p \tilde{x}_p - \zeta_i \tilde{x}_j - \zeta_j \tilde{x}_i - (n-2) \zeta^p \tilde{x}_p \tilde{x}_i \tilde{x}_j \right] \\ & \times \left(|X(\xi_k)|_g + O(\delta_k \|\nabla X\|_{L^\infty(2r_k)}) \right) d_g(\xi_k, x_k)^{1-n} (1 + o(1)), \end{aligned} \quad (9.17)$$

where we have let, up to a subsequence:

$$\tilde{x} = \lim_{k \rightarrow \infty} \frac{1}{d_g(\xi_k, x_k)} \exp_{\xi_k}^{-1}(x_k), \quad K(n) = \frac{n^{\frac{n+2}{2}} (n-2)^{\frac{n}{2}} \omega_n}{2^{n+1} (n-1) \omega_{n-1}} \text{ and } \zeta = \lim_{k \rightarrow \infty} \frac{X(\xi_k)}{|X(\xi_k)|_g}.$$

We conclude this section with a refinement of Proposition 9.2 which takes into account the behavior of v_k “far away” from the concentration point ξ_k .

Proposition 9.3. *Let T_k , Θ_k and T be as in (9.9). Let $(R_k)_k$ be a sequence of positive numbers such that $1 \leq R_k = o(\delta_k^{-\frac{1}{2}})$ as $k \rightarrow +\infty$. There holds, for any $x \in M$:*

$$\begin{aligned} \left| \mathcal{L}_g T_k - \mathcal{L}_g T \right|_g(x) \leq & C \left(\left[|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} \right] \theta_k(x)^{1-n} + \delta_k^{\frac{n}{2}} \theta_k(x)^{1-n} \right. \\ & \left. + \|X\|_{C^0(M)} \varepsilon_k \delta_k R_k^2 + \|X\|_{C^0(M)} \|v_k\|_{L^\infty(M \setminus B_{\xi_k}(R_k \sqrt{\delta_k}))} \right), \end{aligned} \quad (9.18)$$

where C is some positive constant that does not depend on k , on $\|X\|_{C^0(M)}$ and on the choice of $(R_k)_k$.

Proof. Let again G_i , $1 \leq i \leq n$, be the Green fields for $\vec{\Delta}_g$ satisfying (9.3) in $B_{\xi_k}(\delta)$. Mimicking what we did in (9.12), we define:

$$\begin{aligned} Q_k^1(x)_i = & \left(\int_{B_{x_0}(\frac{\delta}{2})} \left\langle G_i(x, y), \left[(u + W_k + v_k)^{2^*} - (u + v_k)^{2^*} \right] (y) X(y) \right\rangle_{g(y)} dv_g(y) \right) \times \eta(d_g(\xi_k, x)), \\ Q_k^2(x)_i = & \left(\int_{B_{x_0}(\frac{\delta}{2})} \left\langle G_i(x, y), \left[(u + v_k)^{2^*} - u^{2^*} \right] (y) X(y) \right\rangle_{g(y)} dv_g(y) \right) \times \eta(d_g(\xi_k, x)). \end{aligned} \quad (9.19)$$

The term $(u + W_k + v_k)^{2^*} - (u + v_k)^{2^*}$ is compactly supported in $B_{\xi_k}(2r_k)$ and satisfies there:

$$\left| (u + W_k + v_k)^{2^*} - (u + v_k)^{2^*} - W_k^{2^*} \right| \leq C \left(W_k^{2^*-1} + W_k \right).$$

Therefore, (9.4) and Giraud’s lemma show that there holds, for any $x \in M$:

$$\left| Q_k^1 \right|_g + \theta_k(x) \left| \mathcal{L}_g Q_k^1 \right|_g \leq \left[|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} \right] \theta_k(x)^{2-n}. \quad (9.20)$$

We thus obtain that there holds:

$$\vec{\Delta}_g (T_k - Q_k^1 - T) = \begin{cases} \left[(u + v_k)^{2^*} - u^{2^*} \right] X & \text{in } B_{\xi_k}(\frac{\delta}{4}), \\ \left[(u + v_k)^{2^*} - u^{2^*} \right] X + O\left(|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} \right) & \text{in } M \setminus B_{\xi_k}(\frac{\delta}{4}). \end{cases} \quad (9.21)$$

We now estimate Q_k^2 in (9.19). With (9.8) we write that there holds, for some positive C :

$$\left| (u + v_k)^{2^*} - u^{2^*} \right| \leq C \times \begin{cases} \varepsilon_k^{2^*} W_k^{2^*} + 1 & \text{in } B_{\xi_k}(\sqrt{\delta_k}), \\ \varepsilon_k & \text{in } B_{\xi_k}(R_k \sqrt{\delta_k}) \setminus B_{\xi_k}(\sqrt{\delta_k}), \\ \|v_k\|_{L^\infty(M \setminus B_{\xi_k}(R_k \sqrt{\delta_k}))} & \text{in } M \setminus B_{\xi_k}(R_k \sqrt{\delta_k}). \end{cases}$$

Then (9.4) and Giraud's lemma yield again, for any $x \in M$:

$$\begin{aligned} |Q_k^2|_g(x) + \theta_k(x) |\mathcal{L}_g Q_k^2|_g(x) &\leq C \left(\varepsilon_k^{2^*} \left[|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} \right] \theta_k(x)^{2-n} + \delta_k^{\frac{n}{2}} \theta_k(x)^{2-n} \right. \\ &\quad \left. + \|X\|_{C^0(M)} \varepsilon_k R_k^2 \delta_k + \|X\|_{C^0(M)} \|v_k\|_{L^\infty(M \setminus B_{\xi_k}(R_k \sqrt{\delta_k}))} \right), \end{aligned} \quad (9.22)$$

so that there holds, with (9.21):

$$\vec{\Delta}_g \left(T_k - Q_k^1 - Q_k^2 - T \right) = \begin{cases} 0 & \text{in } B_{\xi_k}(\delta/4) \\ O \left(|X(\xi_k)|_g + \delta_k \|\nabla X\|_{L^\infty(2r_k)} + \delta_k^{\frac{n}{2}} + \|X\|_{C^0(M)} \varepsilon_k R_k^2 \delta_k \right. \\ \quad \left. + \|X\|_{C^0(M)} \|v_k\|_{L^\infty(M \setminus B_{\xi_k}(R_k \sqrt{\delta_k}))} \right) & \text{in } M \setminus B_{\xi_k}(\delta/4) \end{cases} \quad (9.23)$$

The Claim then follows from (9.23), standard elliptic theory, (9.20) and (9.22). \square

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